BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN

THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER

JOHN R. KLINE HENRY L. RIETZ

VOLUME XLI, NUMBER 6

JUNE, 1935

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1935

Whole No. 429 $9.00 a Year

PUBLISHED MONTHLY

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in the act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R., authorized May 9, 1935.

PERIODICAL PUBLICATIONS
OF THE
AMERICAN MATHEMATICAL SOCIETY

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY

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TETRAHEDRA ASSOCIATED WITH CANONICAL EXPANSIONS FOR A CURVED SURFACE*

BY P. O. BELL

A number of different canonical developments for the equation of a curved surface have been obtained by investigators in the field of projective differential geometry. These developments† are simplest in form when the vertices of associated tetrahedra are located on a certain quadric, known as the canonical quadric of Wilczynski.‡ The geometrical location of this quadric was accomplished by Wilczynski by a very complicated method. Bompiani§ has offered a distinctly different definition, and Stouffer‖, using the general methods of Wilczynski, has found a rather simple method of locating the quadric.

The canonical quadric is actually useful in locating only one of the four vertices of the tetrahedron. It is the purpose of this paper to locate the fourth vertex for the whole series of expansions by rather elementary methods and without the introduction of Wilczynski's quadric. As a matter of fact, the quadric is located as soon as any one of these fourth vertices is determined.

We shall suppose that the asymptotic net is parametric, and take the fundamental differential equations in the form

$$(1) \qquad y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0.$$

Using the notation introduced in the celebrated memoir by Green we shall put

$$(2) \qquad \rho = y_u - \beta y, \qquad \sigma = y_v - \alpha y,$$

where α and β are functions of u and v which may be assigned as desired. If the point y on the surface and the corresponding points ρ and σ are chosen as three vertices of the tetrahedron

* Presented to the Society, April 6, 1935.

† Green, Transactions of this Society, vol. 20 (1919), pp. 79–153.

‡ Wilczynski, Transactions of this Society, vol. 9 (1908), pp. 79–120.

§ Bompiani, Rendiconti Accademia Lincei, (6), vol. 6 (1927), pp. 187–190, and Mathematische Zeitschrift, vol. 29 (1929), pp. 678–683.

‖ Stouffer, Proceedings of the National Academy of Sciences, (18), vol. 3 (1932), pp. 252–255.

of reference, the associated development for the equation of the surface takes its simplest form if the fourth vertex τ is given by

$$\tau = y_{uv} - \alpha y_u - \beta y_v + \alpha\beta y. \tag{3}$$

Under these conditions the point τ is on the quadric of Wilczynski for all values of α and β. Moreover, the line l' joining y and τ is the reciprocal with respect to any quadric of Darboux of the line l joining ρ and σ. Furthermore, the tangent plane at ρ to the ruled surface $R^{(u)}$ generated by the line joining y and ρ as y moves along C_u, (v=const. curve), and the tangent plane at σ to the ruled surface $R^{(v)}$ generated by the line joining y and σ as y moves along C_v, (u=const. curve) intersect in the line l'.

If $\beta = a_u'/(2a')$, $\alpha = b_v/(2b)$, the lines l and l' are the directrices of Wilczynski of the first and second kind, respectively, and if $\beta = -b_u/(4b)$, $\alpha = -a_v'/(4a')$, they are the canonical edges of Green of the first and second kind, respectively. These lines and consequently the associated points ρ and σ are not difficult to locate geometrically. The above stated facts are well known and may be found, for example, in the memoir by Green.*

Let us now consider the set of points

$$\rho_i = y_u - \beta_i y,\ \sigma_j = y_v - \alpha_j y,\ \tau_{ij} = y_{uv} - \alpha_j y_u - \beta_i y_v + \alpha_j\beta_i\, y.$$

The reciprocal lines l and l' determined by ρ_i and σ_j we shall denote by l_{ij} and l_{ij}. In particular we shall denote the directrices of Wilczynski by l_{11} and l_{11}' with the corresponding points denoted by

$$\rho_1 = y_u - \beta_1 y,\ \sigma_1 = y_v - \alpha_1 y,\ \tau_{11} = y_{uv} - \alpha_1 y_u - \beta_1 y_v + \alpha_1\beta_1\, y.$$

Similarly, the canonical edges of Green will be denoted by l_{22} and l_{22}' with the subscript 2 attached to the corresponding points ρ, σ, and τ.

Two points ρ_p, ρ_q and two points σ_r, σ_s determine four lines $l_{pr}, l_{qs}, l_{ps}, l_{qr}$. The lines l_{pr} and l_{qs} intersect in a point $l_{pr}l_{qs}$ which is easily found to be denoted by

$$(\alpha_r - \alpha_s)y_u + (\beta_q - \beta_p)y_v + (\alpha_s\beta_p - \alpha_r\beta_q)y.$$

This point is identical with $\tau_{ps} - \tau_{qr}$. Therefore, the points $l_{pr}l_{qs}$, τ_{ps}, τ_{qr} are collinear. Similarly, the points $l_{qr}l_{ps}$, τ_{qs}, τ_{sr}

* Green, loc. cit.

may be shown to be collinear. Moreover, the above proof holds when $\rho_p \equiv \rho_q$, or when $\sigma_r \equiv \sigma_s$. We therefore have the following theorem.

THEOREM. *The points τ_{ps}, τ_{qr}, and $l_{pr}l_{qs}$ are collinear.*

Now the points τ_{12} and τ_{21} may be located geometrically, as follows. The points ρ and σ for given functions α and β generate two surfaces, s_ρ and s_σ. The tangent at ρ to the curve $u = \text{const.}$ on s_ρ and the tangent at σ to the curve $v = \text{const.}$ on S_σ intersect the line l' in the points $\tau - \beta_v y$ and $\tau - \alpha_u y$, respectively. The harmonic conjugate of y with respect to these two points is the point $\tau - (\alpha_u + \beta_v)y/2$. The harmonic conjugate of the point $\tau - \beta_v y$ with respect to the two points $\tau - \alpha_u y$ and $\tau - (\alpha_u + \beta_v)y/2$ is the point $\tau - (2\alpha_u + \beta_v)y/3$. This is the point τ if α and β are chosen to satisfy the condition

$$2\alpha_u + \beta_v \equiv 0.$$

This condition is satisfied by the selection

$$\alpha = -\frac{a'_v}{4a'}, \qquad \beta = \frac{a'_u}{2a'}.$$

But for this selection $\rho \equiv \rho_1$ and $\rho \equiv \sigma_2$. The point τ for this selection of α and β is therefore the point τ_{12}. The geometric definition of the point τ_{21} is obtained from the definition of τ_{12} by interchanging u and v, ρ and σ, α and β, and replacing a' by b.

It follows from the above theorem that the points $l_{12}l_{11}$, τ_{11}, and τ_{12} are collinear; also, $l_{11}l_{21}$, τ_{11}, and τ_{21} are collinear. Thus τ_{11} is the intersection of the line determined by $l_{12}l_{11}$ and τ_{12} and the line determined by $l_{11}l_{21}$ and τ_{21}, both of which are geometrically located. The point τ_{22} may be located similarly.

We are now in a position to locate any point τ_{rs}. The points $l_{1s}l_{r1}$, τ_{11} and τ_{rs} are collinear, as are also the points $l_{2s}l_{r2}$, τ_{22}, and τ_{rs}. But each of these lines passes through two known points and their intersection determines τ_{rs}. Therefore, upon specifying the locations of the points ρ_r and σ_s, the geometric definition of the point τ_{rs} is complete.

STANFORD UNIVERSITY

ON A THEOREM OF PLESSNER*

BY NELSON DUNFORD

If $f(t)$ is a real-valued function of a real variable, periodic with period 2π and of bounded variation, then $f(t)$ is absolutely continuous provided $v(u)$, the total variation of $f(t+u)-f(t)$ on any interval of length 2π, tends to zero with u. This theorem, which is the converse of a well known theorem of Lebesgue, has been proved by Plessner, and by Wiener and Young.† Ursell‡ has given some interesting results concerning the total variation of $f(t+u)-f(t)$ for measurable functions $f(t)$, which when combined with the above theorem show (as he has pointed out) that that theorem holds for measurable functions. The papers referred to contain the essential ideas sufficient for a very short proof of the general theorem. In fact, with two additional lemmas which are given below, the proof as given by Plessner holds in the general case.

By an *admissible function* will be meant one which is finite-valued and periodic with period 2π.

THEOREM 1. *If $f(t)$ is admissible and $\lim_{u=0} v(u)=0$, then $v(u)$ is continuous.*

Let u_1, u_2 be any real numbers and $\delta_i=(t_{i-1}, t_i)$ be a partition of the interval $(-\pi, \pi)$. Then, if $t_i'=t_i+u_1$, the intervals $\delta_i'=(t_{i-1}', t_i')$ form a partition of $(-\pi+u_1, \pi+u_1)$ and§

$$\delta_i\{f(t+u_1+u_2)-f(t)\} = \delta_i\{f(t+u_1)-f(t)\} + \delta_i'\{f(t+u_2)-f(t)\},$$

and so $v(u_1+u_2)\leqq v(u_1)+v(u_2)$. This shows that v is finite everywhere; for if $v(u)<K$ on $(-a, a)$, then $v(u)<2K$ on $(-2a, 2a)$.

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* Presented to the Society, April 20, 1935.

† Plessner, *Eine Kennzeichnung der totalstetigen Funktionen*, Journal für Mathematik, vol. 160 (1929), pp. 26–32. Wiener and Young, *The total variation of $g(x+h)-g(x)$*, Transactions of this Society, vol. 35 (1933), pp. 327–340.

‡ Ursell, *On the total variation of $f(t+\tau)-f(t)$*, Proceedings of the London Mathematical Society, (2), vol. 37 (1934), pp. 402–415.

§ If $\delta=(a, b)$, by $\delta f(t+u)$ is meant $f(b+u)-f(a+u)$.

From the above inequality it follows that $|v(u)-v(u_1)| \leqq v(u-u_1)+v(u_1-u)$, which shows that $v(u)$ is continuous.

LEMMA 1. *If the function $f(t)$ is admissible and measurable and $\lim_{u=0} v(u)=0$, then $f(t)$ is uniformly continuous.*

If not, there exists a positive number γ such that for every $\alpha>0$ there is an interval $\delta=(t, t')$ in $(-\pi, \pi)$ with $|t-t'|<\alpha$ and such that $|\delta f(t)|>\gamma$. There is an $\alpha_1>0$ such that $v(u)<\gamma/3$ if $|u| \leqq \alpha_1$. Let β be a positive number $<\alpha_1/2$ and $<\gamma/2$. Then by the theorem of Lusin $|f(t)-f(t')|<\beta$ for all t, t' in the interval $I=(-2\pi, 2\pi)$ except those in a set E_β with measure less than β, provided $|t-t'|$ is sufficiently small, say $<\alpha_2$. Fix an interval $\delta=(t, t')$ such that $|t-t'|<\alpha_2$ and $|\delta f(t)|>\gamma$. Then with this δ and any u such that $|u| \leqq \alpha_1$, we have

$$\gamma/3 > v(u) \geqq |\delta\{f(t+u)-f(t)\}| > \gamma - |\delta f(t+u)|.$$

If there exists a u with $|u| \leqq \alpha_1$ such that $t+u$, $t'+u$ are both in $I-E_\beta$, then the above inequality gives the contradiction $\gamma/3>\gamma-\beta>\gamma/2$. To see that such a u exists define a one-to-one correspondence between the intervals $\Delta=(t-\alpha_1, t+\alpha_1)$ and $\Delta'=(t'-\alpha_1, t'+\alpha_1)$ by the equations $x=t+u$, $x'=t'+u$. If E is a set in Δ, E' will denote the corresponding set in Δ'. Now if no such u exists $E_\beta \supset E_\beta\Delta+(\Delta-E_\beta\Delta)'$ and so

$$\begin{aligned}\alpha_1/2 > \beta > m(E_\beta) &\geqq m[E_\beta\Delta + (\Delta - E_\beta\Delta)'] \\ &\geqq [m(E_\beta\Delta) + m(\Delta - E_\beta\Delta)']/2 = \alpha_1,\end{aligned}$$

which again is a contradiction.

THEOREM 2. *If $f(t)$ is admissible and measurable and if also $\lim_{u=0} v(u)=0$, then $f(t)$ is absolutely continuous.*

The proof is entirely analogous to that of Plessner. By Lemma 1, $f(t)$ is summable and if $s_n'(t)$ is the first arithmetic mean of its Fourier series, then

$$s_n'(t) - f(t) = \int_{-\pi}^{+\pi} \{f(t+u) - f(t)\} F_n(u)du,$$

where $F_n(u)$ is Fejér's kernel; and thus the total variation

$$T[s_n'(t) - f(t)] \leqq \int_{-\pi}^{+\pi} v(u)F_n(u)du \to 0$$

by Fejér's theorem, since $v(u)$ is continuous and $v(0)=0$. For $\epsilon>0$ there is an n_0 and a $\delta>0$ such that

$$T[s_{n_0}'(t) - f(t)] < \epsilon/2, \quad \sum |\delta_i s_{n_0}'(t)| < \epsilon/2$$

for an arbitrary set $\{\delta_i\}$ of non-overlapping intervals with $\sum|\delta_i|<\delta$. Thus, for such a set of intervals,*

$$\sum |\delta_i f(t)| \leqq \sum |\delta_i\{f(t) - s_{n_0}'(t)\}| + \sum |\delta_i s_{n_0}'(t)| < \epsilon.$$

It might be pointed out that the theorem as stated is no less general than the corresponding theorem where $f(t)$ is finite and measurable on (a, b) and $v(u)$ is the total variation of $f(t+u) - f(t)$ on $(a, b-u)$.

The following example which is similar to one given by Ursell is interesting in connection with Theorem 2. Let $\{x_\alpha\}$ be a Hamel† base with $x_1=\pi$, so that every real number is expressible uniquely as a finite linear combination of elements of $\{x_\alpha\}$ with rational coefficients. If $t=r_1\pi+\sum r_i x_i$ and $f(t)=r_1-t/\pi$, then $f(t)$ is admissible and additive, that is, $f(t+u)=f(t)+f(u)$, and hence $v(u)$ is identically zero. Theorem 2 shows that $f(t)$ is not measurable and the example shows that the hypothesis of measurability is essential in Theorem 2 as well as in Lemma 1.

BROWN UNIVERSITY

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* My attention is called, by Hans Lewy, to the fact that the use of Fourier series may be avoided by approximating to $f(t)$ by means of the absolutely continuous functions $(1/h)\int_0^h f(t+u)du$.

† Hamel, *Basis aller Zahlen und der unstetigen Lösungen der funktional Gleichungen* $F(x+y)=F(x)+F(y)$, Mathematische Annalen, vol. 60 (1905), p. 459.

ON THE LAW OF QUADRATIC RECIPROCITY*

BY ALBERT WHITEMAN

The following proof of the law of quadratic reciprocity, which depends upon a modified form of the Gaussian criterion, is believed to be new.

According to the usual form of this criterion, if p is any integer not divisible by the odd prime q, then p is a quadratic residue or non-residue of q according as in the series

$$p, 2p, 3p, \cdots, (q-1)p/2,$$

the number of numbers whose least positive remainders (mod q) exceed $q/2$ is even or odd. But, if $\lambda p = \mu q + r$, $q/2 < r < q$, then $2\lambda p = (2\mu+1)q + 2r - q$, and conversely. Hence we have the transformed criterion: p is a quadratic residue or non-residue of q according as the number of least positive odd remainders in the series:

$$2p, 4p, 6p, \cdots, (q-1)p \pmod{q} \tag{1}$$

is even or odd.†

In the following discussion p, q represent any two odd primes such that $q > p$. Let r denote any odd remainder of (1) such that $p < r < q$. Then, for a suitable λ, $(1 \leqq \lambda \leqq (q-1)/2)$,

$$2\lambda p \equiv r \pmod{q}, \tag{2}$$

whence

$$(q + 1 - 2\lambda)p \equiv p + q - r \pmod{q}, \tag{3}$$

where $p < p+q-r < q$.

Congruences (2) and (3) are identical only for $2\lambda = (q+1)/2$, $r = (p+q)/2$. Hence the odd remainders of (1) that are greater than p may be arranged in pairs by means of (2) and (3) except

* Presented to the Society, February 23, 1935.

† For other proofs of the reciprocity law using this transformed criterion see a paper by Lange, *Ein Elementarer Beweis des Reziprozitäts-gesetzes*, Berichte der Koeniglichen Sachsischen Gesellschaft, vol. 48 (1896), p. 629; vol. 49 (1897), p. 607; see also P. Bachmann, *Niedere Zahlentheorie*, Part 1, 1902, pp. 256–261, and pp. 266–267.

when $(q+1)/2$ is even and $(p+q)/2$ is odd, that is, when p, q are each of the form $4n+3$. In this case there is one odd remainder that does not belong to such a pair. If we denote by a the number of odd remainders greater than p, it follows that a is even if at least one of the two primes p, q is of the form $4n+1$, and odd if both are of the form $4n+3$. Consequently

$$a \equiv (p-1)(q-1)/4 \pmod 2. \tag{4}$$

Now let b denote the number of those odd remainders in (1) that are less than p. Then $(p/q)=(-1)^{a+b}$. Also, if c denotes the number of least positive odd remainders in the series

$$2q, 4q, 6q, \cdots, (p-1)q \pmod p, \tag{5}$$

we have $(q/p)=(-1)^c$. Hence

$$(p/q)(q/p) = (-1)^{a+b+c}. \tag{6}$$

To complete the proof, we shall now show that the odd remainders in (1) that are less than p are identical with the odd remainders in (5), and hence that $b=c$. Let

$$2\lambda p \equiv r \pmod q, \tag{7}$$

where now r is an odd remainder such that $0<r<p$, and $1 \leqq \lambda \leqq (q-1)/2$. Hence

$$2\lambda p = (2\mu - 1)q + r,$$

where $0<\mu<(p+1)/2$. From this we obtain

$$(p+1-2\mu)q \equiv r \pmod p. \tag{8}$$

Conversely, from (8), where $1 \leqq \mu \leqq (p-1)/2$, we obtain (7) with $0<\lambda<(q+1)/2$.

Hence, as stated above, the odd remainders in (1) that are less than p are identical with the odd remainders in (5), so that $b=c$. The theorem then follows from (4) and (6).

THE UNIVERSITY OF PENNSYLVANIA

THE ARITHMETIZED EXPANSIONS FOR CERTAIN DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND*

BY G. D. NICHOLS

1. *Introduction.* In this paper the explicit arithmetized Fourier developments are given for certain doubly periodic functions of the third kind, namely,

$$\Phi_{\alpha\beta\gamma}(x, y) = \vartheta_1'^3\vartheta_\alpha(x + y)/\vartheta_\beta^2(x)\vartheta_\gamma(y),$$
$$\Psi_{\alpha\beta\gamma}(x, y) = \vartheta_1'^3\vartheta_\alpha^2(x + y)/\vartheta_\beta^2(x)\vartheta_\gamma^2(y),$$

where x and y are independent complex variables, the ϑ's are the Jacobi theta functions, and α, β, γ are a certain sixteen triads out of the sixty-four which can be selected from the numbers 0, 1, 2, 3. The method of obtaining the expansions is essentially that used by Basoco† in obtaining the arithmetized developments for certain functions similar in form.

The applications to arithmetic lead to arithmetical paraphrase equations involving incomplete numerical functions in two variables. The complexity of the expansions is not adverse to simple and interesting arithmetical results, since simplicity can be gained by specialization while the complexity gives greater variety.

There are thirty-two expansions given in all. The domain of validity for x is $-I(\pi\tau) < I(x) < I(\pi\tau)$ in (1)–(8) and (17)–(24) inclusive, and $-(1/2)I(\pi\tau) < I(x) < (1/2)I(\pi\tau)$ for (9)–(16) and (25)–(32) inclusive; y is restricted in all expansions to $-(1/2)I(\pi\tau) < I(y) < (1/2)I(\pi\tau)$.

In all the series the outer summation refers to positive non-zero integers m, n, α in the exponent of q, and extends to all $m = 1, 3, 5, \cdots$, to all $n = 1, 2, 3, \cdots$, or $2, 4, 6, \cdots$, to all $\alpha = 1, 5, 9, \cdots$. The coefficient of the power of q is in parentheses after the power. The inner summation is finite and refers to positive integral divisors t, τ, d, δ, of m, n, α, subject to further restrictions noted below. The integer τ is always odd. The

* Presented to the Society, November 30, 1934.

† American Journal of Mathematics, vol. 54 (1932), p. 242.

values and necessary arithmetized expansions for the coefficients A_i, B_i, C_i, $(i=0, 1, 2, 3)$, are given below in §4.

2. *The Functions* $\Phi_{\alpha\beta\gamma}(x, y)$. In this section $N1$, $N2$, $\cdots$, where $1, 2, \cdots$ refer to the serial number of the expansion, are equal to the expression in brackets following $A_i/2$. The symbol $\mathcal{D}$ is the operator $(\partial/\partial x+2\partial/\partial y)$.

In the expansions (1)–(4) inclusive, we have $n=d\delta$, $d\leqq n^{1/2}$, $z=2(\delta-d)x+2dy$. When $d=n^{1/2}$ the value obtained from the inner summation is to be divided by 2.

$$(1)\quad \Phi_{111}(x, y) = (1/2)A_1[\operatorname{ctn} x + 4\sum q^{2n}(\sum \sin z)] - C_1\mathcal{D}(N1) + (1/2)\mathcal{D}^2(N1),$$

$$(2)\quad \Phi_{221}(x, y) = (1/2)A_1[-\tan x + 4\sum q^{2n}(\sum (-1)^{\delta-d}\sin z)] - C_1\mathcal{D}(N2) + (1/2)\mathcal{D}^2(N2),$$

$$(3)\quad \Phi_{212}(x, y) = (1/2)A_2[\operatorname{ctn} x + 4\sum q^{2n}(\sum (-1)^{d}\sin z)] - C_2\mathcal{D}(N3) + (1/2)\mathcal{D}^2(N3),$$

$$(4)\quad \Phi_{122}(x, y) = (1/2)A_2[\tan x - 4\sum q^{2n}(\sum (-1)^{\delta}\sin z)] - C_2\mathcal{D}(N4) + (1/2)\mathcal{D}^2(N4).$$

In (5)–(8) inclusive $n=t\tau$, $t<(n/2)^{1/2}$, $z=(\tau-2t)x+2ty$.

$$(5)\quad \Phi_{010}(x, y) = (1/2)A_0[\csc x + 4\sum q^{n}(\sum \sin z)] - C_0\mathcal{D}(N5) + (1/2)\mathcal{D}^2(N5),$$

$$(6)\quad \Phi_{320}(x, y) = (1/2)A_0[\sec x + 4\sum q^{n}(\sum(-1/\tau - 2t)\cos z)] - C_0\mathcal{D}(N6) + (1/2)\mathcal{D}^2(N6),$$

$$(7)\quad \Phi_{313}(x, y) = (1/2)A_3[\csc x + 4\sum q^{n}(\sum (-1)^{t}\sin z)] - C_3\mathcal{D}(N7) + (1/2)\mathcal{D}^2(N7),$$

$$(8)\quad \Phi_{023}(x, y) = (1/2)A_3[\sec x + 4\sum q^{n}(\sum (-1/\tau)\cos z)] - C_3\mathcal{D}(N8) + (1/2)\mathcal{D}^2(N8).$$

In (9)–(12) inclusive $m=t\tau$, $\tau\leqq m^{1/2}$, $z=(t-\tau)x+\tau y$. When $\tau=m^{1/2}$ the value obtained from the inner summation is to be divided by 2.

$$(9)\quad \Phi_{001}(x, y) = (1/2)A_1[-4\sum q^{m/2}(\sum \sin z)] - C_1\mathcal{D}(N9) + (1/2)\mathcal{D}^2(N9),$$

$$(10)\quad \Phi_{331}(x, y) = (1/2)A_1[-4\sum q^{m/2}(\sum (-1)^{(t-\tau)/2}\sin z)] - C_1\mathcal{D}(N10) + (1/2)\mathcal{D}^2(N10),$$

$$(11)\quad \Phi_{302}(x, y) = (1/2)A_2[-4\sum q^{m/2}(\sum (-1/\tau)\cos z)] - C_2\mathcal{D}(N11) + (1/2)\mathcal{D}^2(N11),$$

$$(12)\quad \Phi_{032}(x, y) = (1/2)A_2[4\sum q^{m/2}(\sum (-1/\tau)\cos z)] - C_2\mathcal{D}(N12) + (1/2)\mathcal{D}^2(N12).$$

In (13)–(16) inclusive $n=t\tau$, $\tau<n^{1/2}$, $z=(t-\tau)x+\tau y$, t assuming even positive integral values only.

$$(13)\quad \Phi_{203}(x, y) = (1/2)A_3[-4\sum q^{n/2}(\sum (-1/\tau)\cos z)] - C_3\mathcal{D}(N13) + (1/2)\mathcal{D}^2(N13),$$

$$(14)\quad \Phi_{133}(x, y) = (1/2)A_3[4\sum q^{n/2}(\sum (-1)^{t/2}\sin z)] - C_3\mathcal{D}(N14) + (1/2)\mathcal{D}^2(N14),$$

$$(15)\quad \Phi_{100}(x, y) = (1/2)A_0[-4\sum q^{n/2}(\sum \sin z)] - C_0\mathcal{D}(N15) + (1/2)\mathcal{D}^2(N15),$$

$$(16)\quad \Phi_{230}(x, y) = (1/2)A_0[4\sum q^{n/2}(\sum (-1/\tau + t)\cos z)] - C_0\mathcal{D}(N16) + (1/2)\mathcal{D}^2(N16).$$

3. *The Functions* $\Psi_{\alpha\beta\gamma}(x, y)$. Here $N17$, $N18$, $\cdots$ are equal to the expressions in brackets following B_i, $\mathcal{D}$ is the operator $(2\partial/\partial x+\partial/\partial y)$. The substitution (i, j) is one which replaces B_i and C_i in the right member of any expansion by B_j and C_j, respectively.

In (17)–(24) inclusive $n=d\delta$, $\delta-d\equiv 1 \pmod 2$, $d<n^{1/2}$, $z=(\delta-d)x+4dy$.

$$(17)\quad \Psi_{111}(x, y) = B_1[\csc x + 4\sum q^n(\sum (-1)^d \sin z)] - C_1\mathcal{D}(N17) + (1/8)\mathcal{D}^2(N17),$$

$$(18)\quad \Psi_{221}(x, y) = B_1[\sec x + 4\sum q^n(\sum (-1/d+\delta)\cos z)] - C_1\mathcal{D}(N18) + (1/8)\mathcal{D}^2(N18),$$

(19) $\Psi_{212}(x, y)$ is written from (17) by the substitution (1, 2),

(20) $\Psi_{122}(x, y)$ is written from (18) by the substitution (1, 2),

(21) $\Psi_{010}(x, y)$ is written from (17) by the substitution (1, 0),

(22) $\Psi_{320}(x, y)$ is written from (18) by the substitution (1, 0),

(23) $\Psi_{313}(x, y)$ is written from (17) by the substitution (1, 3),

(24) $\Psi_{023}(x, y)$ is written from (18) by the substitution (1, 3).

In the formulas (25)–(32) inclusive $\alpha = t\tau$, $\alpha \equiv 1 \pmod 4$, $\tau \leqq \alpha^{1/2}$, $z = ((t-\tau)/2)x + 2\tau y$. When $\tau = \alpha^{1/2}$ the value obtained from the inner summation is to be divided by 2.

(25) $$\Psi_{001}(x, y) = B_1[4\sum q^{\alpha/4}(\sum (-1/\tau)\cos z)] - C_1\mathfrak{D}(N25) + (1/8)\mathfrak{D}^2(N25),$$

(26) $$\Psi_{331}(x, y) = B_1[4\sum q^{\alpha/4}(\sum (-1)^{(t+\tau-2)/4}\cos z)] - C_1\mathfrak{D}(N26) + (1/8)\mathfrak{D}^2(N26),$$

(27) $\Psi_{302}(x, y)$ is written from $-\Psi_{001}(x, y)$ by the substitution (1, 2),

(28) $\Psi_{032}(x, y)$ is written from $-\Psi_{331}(x, y)$ by the substitution (1, 2),

(29) $\Psi_{203}(x, y)$ is written from $-\Psi_{001}(x, y)$ by the substitution (1, 3),

(30) $\Psi_{133}(x, y)$ is written from $-\Psi_{331}(x, y)$ by the substitution (1, 3),

(31) $\Psi_{100}(x, y)$ is written from (25) by the substitution (1, 0),

(32) $\Psi_{230}(x, y)$ is written from (26) by the substitution (1, 0).

4. *Arithmetization of the Coefficients.* The values of the coefficients are

$$A_i = \frac{\vartheta_i''(y)}{\vartheta_i(y)} - \frac{\vartheta_1'''}{\vartheta_1'}, \quad B_i = \frac{\vartheta_i''(y)}{\vartheta_i(y)} + \frac{\vartheta_i'^2(y)}{\vartheta_i^2(y)} - \frac{\vartheta_1'''}{2\vartheta_1'}, \quad C_i = \frac{\vartheta_i'(y)}{\vartheta_i y}.$$

The necessary arithmetized expansions for the C_i's are given in a list by Bell.* In the following expansions for the others, two types of divisors occur, namely, $n = d\delta$, where d is arbitrary and δ even, and $n = t\tau$, where t is arbitrary and τ odd.

(a) $$A_0 = 1 + 8\sum q^n(\sum \tau \cos 2ty) - 8\sum q^n(\sum \delta),$$

(b) $$A_1 = 8\sum q^n(\sum \delta \cos 2dy) - 8\sum q^n(\sum \delta),$$

* Messenger of Mathematics, vol. 53 (1924), p. 166.

$$(c)\quad A_2 = 8\sum q^n\Big(\sum (-1)^d\delta \cos 2dy\Big) - 8\sum q^n\Big(\sum \delta\Big),$$

$$(d)\quad A_3 = 1 + 8\sum q^n\Big(\sum (-1)^t\tau \cos 2ty\Big) - 8\sum q^n\Big(\sum \delta\Big),$$

$$(e)\quad B_0 = 1/2 + 2\sum q^n\Big(\sum \delta\Big) + 8\sum q^n\Big(\sum (2\tau - t)\cos 2ty\Big),$$

$$(f)\quad B_1 = -3/2 + \csc^2 y + 2\sum q^n\Big(\sum \delta\Big) + 8\sum q^n\Big(\sum (2\delta - d)\cos 2dy\Big),$$

$$(g)\quad B_2 = -3/2 + \sec^2 y + 2\sum q^n\Big(\sum \delta\Big) + 8\sum q^n\Big(\sum (2\delta - d)(-1)^d \cos 2dy\Big),$$

$$(h)\quad B_3 = 1/2 + 2\sum q^n\Big(\sum \delta\Big) + 8\sum q^n\Big(\sum (-1)^t(2\tau - t)\cos 2ty\Big).$$

UNIVERSITY OF ARKANSAS

A CONNECTEDNESS THEOREM IN ABSTRACT SETS*

BY W. M. WHYBURN

This note gives a variation of a theorem of Sierpinski and Saks.† The theorem is valid in spaces which have the Borel-Lebesgue property (Axiom I of Saks‡) and which satisfy axioms (A), (B), (C), and (6) as given by Hausdorff.§ We use the term *connected* for a closed set to mean that the set cannot be expressed as the sum of two mutually exclusive non-vacuous, closed sets.‖

THEOREM. *Let F be a collection of closed sets at least one of which is compact. Let F contain more than one element and let it be true that the sets of each finite sub-collection of F have a non-vacuous, connected set in common when this sub-collection contains at least two elements of F. Under these hypotheses, there is a closed, non-vacuous, connected set common to all of the sets of collection F.*

PROOF. Let F_0 be a compact member of collection F and let K be the set of points common to all of the sets of collection F.

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* Presented to the Society, December 1, 1934.

† See Saks, Fundamenta Mathematicae, vol. 2 (1921), pp. 1–3.

‡ Saks, ibid., p. 2.

§ *Mengenlehre*, 1927, pp. 228–229.

‖ The notion of *limit point* may be defined and this definition used to describe connectedness. We use *domain* and *open set* interchangeably.

Saks* shows that K is non-vacuous while the closure of K is an immediate consequence of the closure of the sets of collection F (since any point of the complement of K has a neighborhood which belongs to the complement of some one of the sets of collection F and hence belongs to the complement of K). It remains to show that K is connected. Suppose $K=K_1+K_2$, where K_1 and K_2 are mutually exclusive, non-vacuous, closed sets. The collection C composed of the complements of the sets of collection F is a set of domains that covers F_0-K. By axiom (6), for each point p of K_1 there exist mutually exclusive domains G_{1p} and G_{2p} such that $p\epsilon G_{1p}$, $K_2 \subseteq G_{2p}$. Let $[G_{1p}]$ and $[G_{2p}]$ be the collections of domains obtained in this manner for all points of K_1. The set K_1 is closed and compact and hence has the Borel-Lebesgue property. Let $G_1, \cdots, G_n$ be a finite sub-collection of $[G_{1p}]$ which covers K_1 and let $H_1, \cdots, H_n$ be the corresponding members of $[G_{2p}]$. If H denotes the common part of $H_1, \cdots, H_n$, then H is a domain that covers K_2 (this follows from a theorem stated by Hausdorff, loc. cit., page 229, line 4) while $G=G_1+G_2+\cdots+G_n$ is a domain that covers K_1. Furthermore, H and G have no point in common since G_i and H_i are mutually exclusive sets. The collection C together with G and H cover the closed and compact set F_0. The Borel-Lebesgue property yields a finite collection $C_1, C_2, \cdots, C_m, G, H$, of these sets that covers F_0 while the hypotheses of the theorem together with the method of construction of the covering sets force this collection to contain G, H, and at least one of the sets C_i. Let F_i be the complement of C_i and let Q be the set common to $F_0, F_1, \cdots, F_m$. The set Q contains K_1 and K_2 and is covered by G and H (since Q belongs to F_0 and the complements of $C_1, C_2, \cdots, C_m$). Since G and H are mutually exclusive, it follows that Q is not connected. This contradicts the hypothesis that any finite collection of two or more of the sets of F has a connected set in common and yields the theorem.

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES

* Loc. cit., p. 2.

NOTE ON NON-ANALYTIC FUNCTIONS*

I. M. SHEFFER

In his retiring presidential address before the American Mathematical Society (December, 1931), E. R. Hedrick gave a resumé of the theory of non-analytic functions of a complex variable.† We propose in the present note to establish, with regard to these functions, some elementary properties that do not appear in the Hedrick report. (In the course of the work, however, some properties considered by Hedrick will present themselves.)

Let

$$w = u + iv = f(z), \qquad z = x + iy, \tag{1}$$

where u, v have continuous first partial derivatives in x and y, in some plane region $\mathfrak{R}$. Equation (1) can be expressed as a point transformation in the plane:

$$T: \quad u = u(x, y), \qquad v = v(x, y). \tag{2}$$

If f is non-analytic in $\mathfrak{R}$ (as we shall assume throughout), then T is not a *directly conformal* transformation; that is, the magnitude and sense of angles are not both preserved under T. Let m_1, m_2 be (the slopes of) any two directions at a point $z=x+iy$. In general they will transform into directions not forming the same angle (magnitude and sense both considered). In fact, the following theorem is easily verified by elementary methods.

THEOREM 1. *A necessary and sufficient condition that two distinct directions m_1, m_2 at a point $z=x+iy$ transform conformally‡ under T is that m_1 and m_2 satisfy the relation*

$$(G - J)m_1m_2 + F(m_1 + m_2) + (E - J) = 0. \tag{3}$$

Here

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* Presented to the Society, April 18, 1930.

† *Non-analytic functions of a complex variable*, this Bulletin, vol. 39 (1933), pp. 75–96.

‡ By *conformal* we shall mean *directly conformal*.

$$(4)\qquad E=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2,\qquad F=\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y},$$
$$G=\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2,\qquad J=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}.$$

Among the quantities E, F, G, J there are some easily established identities, of which the following are of use to us:

$$(5)\qquad J^2=EG-F^2,$$

$$(6)\qquad (J-E)\frac{\partial v}{\partial y}+F\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+(J-G)\frac{\partial u}{\partial x}=0,$$

$$(7)\qquad (J-E)\frac{\partial u}{\partial y}+F\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-(J-G)\frac{\partial v}{\partial x}=0.$$

A point (x, y) may be such that all angles at the point transform conformally. A (necessary and sufficient) condition for this is that the Cauchy-Riemann equations be satisfied at the point:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

We shall consider only such points as *do not have the above property*.

COROLLARY. *With each direction m_1 at $z=x+iy$, there is associated a unique direction m_2, which we term the dual direction, such that the angle formed by m_1, m_2 transforms conformally. If m_2 is the dual of m_1, then m_1 is the dual of m_2.*

Of the pairs of dual directions m_1, m_2 at a point, three have special significance: *self-dual* or *contact* directions (of which there are two) and *ortho* directions.* These are defined as dual directions which are, respectively, coincident and orthogonal; that is, for which, respectively, $m_2=m_1$, and $m_2=-1/m_1$. Referring to (3), we have (on setting $m=dy/dx$) the following theorem.

THEOREM 2. *The contact directions and the ortho directions are defined respectively by*

* The ortho directions are termed *principal directions* by Hedrick, loc. cit., p. 78, and were first found by Tissot.

$$(G - J)dy^2 + 2Fdydx + (E - J)dx^2 = 0, \tag{8}$$

$$Fdy^2 + (E - G)dydx - Fdx^2 = 0. \tag{9}$$

As (x, y) varies, (8) and (9) define two systems of curves, which we may term the system of *contact curves* and the system* of *characteristic curves*. The discriminant of (9) is non-negative.

COROLLARY. *The characteristic curves form a real orthogonal system.*

On the other hand, the contact curves need not be real. In fact, the discriminant of (8) is

$$4J(E + G - 2J) = 4J\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right].$$

COROLLARY. *The contact curves are real at (and only at) those points (x, y) where $J \geqq 0$.*

The function $f(z)$ being non-analytic, a unique derivative does not exist. Rather, *in each direction m* at a point, there is a unique derivative. One of the forms in which it can be written is

$$\frac{df}{dz_m} = \frac{1}{(1 + m^2)}\left[\left\{\frac{\partial u}{\partial x} + m\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + m^2\frac{\partial v}{\partial y}\right\} + i\left\{\frac{\partial v}{\partial x} + m\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) - m^2\frac{\partial u}{\partial y}\right\}\right]. \tag{10}$$

Kasner has shown that† as m takes on all values, df/dz_m traces out a circle, termed by Hedrick the Kasner circle for the point (x, y). From (10) we obtain‡

$$\left|\frac{df}{dz_m}\right| = \left\{\frac{1}{(1 + m^2)}(E + 2Fm + Gm^2)\right\}^{1/2}, \tag{11}$$

$$\operatorname{amp}\left(\frac{df}{dz_m}\right) = \tan^{-1}\left\{\frac{\dfrac{\partial v}{\partial x} + m\left(\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial x}\right) - m^2\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial x} + m\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) + m^2\dfrac{\partial v}{\partial y}}\right\}. \tag{12}$$

* For equation (9) and the latter system, see Hedrick, loc. cit., p. 78.

† Hedrick, loc. cit., pp. 80–81.

‡ See Hedrick, loc. cit., p. 78, for (11).

Now from (8), $E+2Fm+Gm^2=J(1+m^2)$. Hence we have the following theorem.

THEOREM 3. *In a contact direction m,*

$$\left|\frac{df}{dz_m}\right| = |J|^{1/2}. \tag{13}$$

If we apply the methods of the differential calculus to (11) and (12), we obtain the following result.

THEOREM 4. *The extremal values of $|df/dz_m|$ are* in the ortho (that is, the principal) directions, and the extremal values of* amp (df/dz_m) *are in the contact directions (when these are real).*

Denote the Kasner circle (for a given point (x, y)) by K. From the extremal properties of systems (8) and (9) we obtain the following theorem.

THEOREM 5. *If O is the origin and Z the center of the Kasner circle K, then the line through O and Z cuts K in points which give df/dz_m in the characteristic directions, and the points of tangency of the tangents to K from O (when O is not interior to K) give df/dz_m in the contact directions.*

From Theorems 3 and 5 we obtain the following corollary.

COROLLARY. *The common length of the tangents† from O to K (when they exist) is $|J|^{1/2}$.*

Let m_1, m_2 be a pair of dual directions. We may ask where the corresponding points df/dz_{m_1}, df/dz_{m_2} lie on K. The answer is given by the following theorem.

THEOREM 6. *Let m_1, m_2 be dual directions. Then the points df/dz_m for $m=m_1$, m_2 lie (on K) on a line through the origin. Conversely, points on K that are concurrent with O correspond to dual directions.*

To establish this, let us equate tan $\{\text{amp}(df/dz_m)\}$ to a constant k, using (12). This yields the equation

* See Hedrick, loc. cit., p. 78.

† See Hedrick, loc. cit., p. 84.

$$m^2\left[k\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\right]+m\left[k\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right]$$
$$+\left[k\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right]=0.$$

Denoting the two roots by m_1 and m_2, we have

$$m_1+m_2=\frac{-\left[k\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right]}{k\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}},$$

$$m_1m_2=\frac{k\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}}{k\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}}.$$

If now we substitute these values in (3), we find, on using (6) and (7), that (3) is satisfied identically for all k. Hence the converse is true. But all points of K are thus paired off, and since a given direction has but *one* dual, the first half of the theorem is likewise true.

For all lines through O, meeting the Kasner circle K in points M_1, M_2, the product $OM_1\cdot OM_2=$constant. That constant must, by the Corollary* to Theorem 5, be $|J|$. Hence from Theorem 5 we deduce the next theorem.

THEOREM 6. *If m_1, m_2 are any two dual directions, then*

$$\left|\frac{df}{dz_{m_1}}\cdot\frac{df}{dz_{m_2}}\right|=|J|. \tag{14}$$

If we set $m=\tan\theta$, the Kasner circle can be written parametrically as follows:†

$$\frac{df}{dz_m}=\mathfrak{D}[f]+\mathfrak{P}[f]e^{-2i\theta}, \tag{15}$$

* Strictly speaking, the Corollary applies only when O is not interior to K; but the result can be shown to be true in all cases.

† Hedrick, loc. cit., p. 76.

where

$$\mathcal{D}[f] = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right],$$

$$\mathcal{P}[f] = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]. \tag{16}$$

Kasner has termed the quantity $\mathcal{D}[f]$, which is the center of the circle, the *mean derivative of f;* and indeed it is an average of df/dz_m for all points of K. There is a second point, $\mathcal{D}^*[f]$, which we believe may prove of use. It is obtained as follows. If we pair off points on K corresponding to dual directions, their (geometric) mean absolute value is $|J|^{1/2}$ (Theorem 6). Such a pair of points lie on a line through O, and if we average, arithmetically, the amplitudes (that is, inclinations) of such lines, we obtain the amplitude of the line through the center. We may therefore consider the quantity

$$\mathcal{D}^*[f] = |J|^{1/2} e^{i\alpha}, \tag{17}$$

where

$$\alpha = \operatorname{amp}\mathcal{D}[f], \quad \tan\alpha = \frac{v_x - u_y}{u_x + v_y},$$

as being, in some sense, an average derivative. A simple reduction gives us

$$\mathcal{D}^*[f] = \frac{2|J|^{1/2}}{(E + G + 2J)^{1/2}}\mathcal{D}[f]. \tag{18}$$

$\mathcal{D}^*[f]$ is thus defined for all points (x, y), where $E+G+2J \neq 0$. Now $E+G+2J=0$ when and only when $\mathcal{D}[f]=0$; which is to say, if and only if the point (x, y) is one at which the transformation (2) is *inversely-conformal*. Hence $\mathcal{D}^*[f]$ is defined whenever the K-circle does not have its center at the origin. And when the center is at the origin, $\mathcal{D}^*[f]$ is indeterminate.

PENNSYLVANIA STATE COLLEGE

A NEW SOLUTION OF THE GAUSS PROBLEM ON $h(s^2d)/h(d)$*

BY GORDON PALL

The following demonstration of the well known formula

$$h(p^2d') = \sigma^{-1}\{p - (d' \mid p)\}h(d') \tag{1}$$

may be worth noting. Here $h(\Delta)$ denotes the number of classes of primitive integral binary quadratic forms of non-zero discriminant Δ; p is any prime $\geqq 2$; $\sigma = 1$ if $d' < -4$ or d' is a square, $\sigma = 2$ if $d' = -4$, $\sigma = 3$ if $d' = -3$; and if d' is positive but not square, σ is the least positive integer for which $p \mid u_\sigma$, (t_k, u_k) denoting the successive positive integral solutions of $t^2 - d'u^2 = 4$.

Let $r(n)$ denote the number of sets of representations of n by a representative system of primitive forms of discriminant $d = p^2d'$. If q is a prime such that $(d \mid q) = 1$,

$$r(p^2q) = 2\{p - (d' \mid p)\}. \tag{2}$$

For by II (5), (33), (23)–(24),†

$$r(p^2q) = r(p^2)r(q) = 2r(p^2) = 2\{1 + r'(p^2)\},$$

where $r'(p^2)$ equals the number $p - 1 - (d' \mid p)$ of solutions w of

$$(pw)^2 \equiv p^2d' \pmod{4p^2}, \quad \frac{w^2 - d'}{4} \text{ prime to } p, \quad (0 \leqq pw < 2p^2).$$

By Theorem 4 of I, extended to $d > 0$ in II, there is associated with each class (connoted by K, say) of primitive forms f of discriminant p^2d', a unique ambiguous class C, or two non-ambiguous classes C and C^{-1}, of primitive forms g of discriminant d'; C is characterized as representing any prime represented by K. By II (13), such forms satisfy, for all integers n,

$$f(p^2n) = \sigma g(n). \tag{3}$$

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* Presented to the Society, April 6, 1935.

† References are to the writer's two papers: I, Mathematische Zeitschrift, vol. 36 (1933), pp. 321–343; and II, Transactions of this Society, vol. 35 (1933), pp. 491–509.

Choose n to be a prime q represented by C and prime to d. Then $g(q)=2$ if C is ambiguous, $g(q)=1$ if $C\neq C^{-1}$. If a form f_1 is associated with a form g_1 not in C or C^{-1}, $f_1(p^2q)=\sigma g_1(q)=0$. Hence, by (2) and (3), p^2q is represented in exactly $\eta\{p-(d'|p)\}\sigma^{-1}$ classes K, where η is 1 or 2 according as q is represented in only one (ambiguous) or two (reciprocal) primitive classes of discriminant d'.

McGill University

ON A REDUCTION OF A MATRIX BY THE GROUP OF MATRICES COMMUTATIVE WITH A GIVEN MATRIX*

BY P. L. TRUMP

1. *Introduction.* Two $n\times n$ matrices A and B, with elements in any field F, are said to be similar in F if there exists a non-singular $n\times n$ matrix S, with elements in F, such that $S^{-1}AS=B$.

Ingraham† has given a method for finding the most general solution, with elements in F, of the matrix equation

$$P(X) = A,$$

where $P(X)$ is a polynomial with coefficients in F, and A is a square matrix with elements in F. A certain set of dissimilar solutions $X_1, X_2, \cdots, X_k$ were obtained in terms of which the complete system of solutions was seen to be in the form $S^{-1}X_iS$, where S is commutative with A. The X_i's are obviously commutative with A.

The purpose of this investigation is to determine the conditions under which two $n\times n$ matrices C and D are similar under transformations of the group $[S]$ of non-singular matrices S which are commutative with a certain $n\times n$ matrix A, where the matrices C and D are also commutative with A. We then seek to describe possible canonical forms to which such matrices

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* Presented to the Society, September 4, 1934. This paper with proofs and detail that are omitted here, is on file as a doctor's thesis at the Library of the University of Wisconsin.

† *On the rational solutions of the matrix equation* $P(X)=A$, Journal of Mathematics and Physics, vol. 13 (1934), pp. 46–50.

may be reduced. At this writing the reduction has been completed with the exception of one case, the nature of which will be made clear. Procedures may in any case be given, however, for determining whether or not two particular matrices are similar under the group $[S]$. Such procedures lead to the necessity of solving a set of linear equations, a procedure which becomes involved and fails to give general results.

2. *Matrices Commutative with A.* Let T be any non-singular square matrix. The condition that $AC=CA$ is equivalent to the condition that $GH=HG$, where $G=T^{-1}AT$ and $H=T^{-1}CT$. Thus A may be considered in classical canonical form consisting of zeros except for square matrices A_i, $(i=1, 2, \cdots, r)$, along the main diagonal. The elements of A_i are zeros except for the characteristic roots α_i (not necessarily in F) of A on the main diagonal and possibly 1's in certain positions on the first diagonal below. Further, consider α_i different from α_j, if i is different from j.

The matrix C is commutative with A if and only if it is zero except for principal minor matrices C_i of the same dimension and in the same position as A_i, $(i=1, 2, \cdots, r)$, where the form of C_i is determined by the condition that $AC=CA$ if and only if $A_iC_i=C_iA_i$. Further, if S is also in the form just referred to, then $S^{-1}CS=D$ is again of the same form with $S_i^{-1}C_iS_i=D_i$.

We are thus led to a consideration of the case in which all the characteristic values of A are the same, in particular, we may assume them zero. The elementary divisors of A are then

$$\lambda^{n_1}, \lambda^{n_2}, \cdots, \lambda^{n_m},$$

where we may assume $n_i \geqq n_{i+1}$, $(i=1, 2, \cdots, m-1)$.

We will describe the frequently displayed form of the most general matrix C commutative with A in this form:*

Case I. If A has the single elementary divisor λ^{n_1}, then C is an $n_1 \times n_1$ diagonalized matrix (c_{ij}), where c_{ij} is arbitrary except for the conditions

$$c_{ij}=0, \quad (i<j), \quad c_{ij}=c_{i+1,j+1}, \quad (i, j=1, 2, \cdots, n_1-1).$$

Case II. In the more general case that A has m elementary divisors we will consider C to be a matrix of matrices C_{ij},

* Cullis, *Matrices and Determinoids*, vol. 3, part 1, sec. 242.

$(i, j=1, 2, \cdots, m)$, where the block C_{ii}, of order n_i, is as described in Case I. The block C_{ij}, $i<j$, will be an $n_i \times n_j$ matrix in which the first n_i-n_j rows are zero and the remaining $n_j \times n_j$ matrix is of the same form as C_{jj}. If $i>j$, C_{ij} will be an $n_i \times n_j$ matrix in which the last n_j-n_i columns are zero and the remaining $n_i \times n_i$ matrix is of the same form as C_{ii}. Define the ρ's as follows, if $i<j$: $\rho_{ij}=\rho_{jj}=n_i-n_j$, $\rho_{ii}=0$. The elements of C_{ij} will be indicated by c_{ijk}. If the element occurs in the rth diagonal of the indicated square portion of C_{ij}, then $k=2(r-1)+\rho_{ij}$.

In order to illustrate, let the elementary divisors of A be λ^3 and λ^2. We will then assume A and C in the following forms

$$A = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix}, \quad C = \begin{Vmatrix} c_{110} & 0 & 0 & 0 & 0 \\ c_{112} & c_{110} & 0 & c_{121} & 0 \\ c_{114} & c_{112} & c_{110} & c_{123} & c_{121} \\ c_{211} & 0 & 0 & c_{220} & 0 \\ c_{213} & c_{211} & 0 & c_{222} & c_{220} \end{Vmatrix}.$$

We may now define certain unit matrices in terms of which C may be expressed. Let e_{ijk} represent the matrix of the same dimension as C with $c_{ijk}=1$ and all other elements zero. We may then write C in the form

$$C = \sum_{s,\,t,\,t} c_{ijt} e_{ijt}, \qquad (i, j = 1, 2, \cdots, m), \tag{1}$$

and $t-\rho_{ij}=0, 2, \cdots, 2(n_1-1)$, where l is the greater of i and j. The multiplication table for these basal units e_{ijk} is as follows:

$$e_{ijk}e_{rst} = \begin{cases} 0, & j \neq r, \\ e_{i,s,k+t}, & j = r. \end{cases}$$

If $k+t>2(n_l-1)+\rho_{ij}$, where l is the greater of i and s, then $e_{i,s,k+t}=0$.

3. *An Isomorphism.* At this point we will establish an isomorphism through which we are led to a consideration of matrices of reduced dimension whose elements are certain polynomials corresponding to the blocks C_{ij}. To C as expressed in (1) we make correspond a matrix P of the form

$$P = (P_{ij}(\gamma^2)\gamma^{\rho_{ij}}), \qquad (i, j = 1, 2, \cdots, m), \tag{2}$$

where

$$P_{ij}(\gamma^2) = c_{ijw} + c_{ijw+2}\gamma^2 + c_{ijw+4}\gamma^4 + \cdots,$$

with $w=\rho_{ij}$. The product of P and $Q=(Q_{ij}(\gamma^2)\gamma^{\rho_{ij}})$ is determined as usual and then the polynomial in the ith row and jth column is reduced modulo $\gamma^{2(n_l-1)+\rho_{ij}}$, where l is the greater of i and j.

The correspondence is obviously preserved under addition and scalar multiplication. To establish the preservation of the correspondence under multiplication, it is convenient to consider all $m\times m$ matrices of the type P with γ^k in the ith row and jth column, where $k-\rho_{ij}=0, 2, \cdots, 2(n_l-1)$, ($l$ the greater of i and j), and zeros elsewhere, as a set of basal units in terms of which matrices of type P may be expressed with scalar coefficients. Indicate such a matrix by f_{ijk}. Let e_{ijk} correspond to f_{ijk}. Since the multiplication table for these basal units is the same in each case the isomorphism is established.

It can be shown that the determinant of P is a polynomial in γ^2 and that P^{-1} exists and is of type (2) if and only if $|P| \not\equiv 0 \bmod \gamma$.

4. *Reduction to a Semi-Canonical Form.* We proceed with the reduction of matrices defined by the isomorphism by transformations with non-singular matrices of the same type. Consider

$$Q = (Q_{ij}(\gamma^2)\gamma^{\rho_{ij}}),$$

where

$$Q_{ij}(\gamma^2) = q_{ijt} + q_{ijt+2}\gamma^2 + \cdots, \qquad (t = \rho_{ij}).$$

Suppose

$$\rho_{uv} = 0, \text{ but } \rho_{u-1,u} \neq 0 \text{ and } \rho_{v,v+1} \neq 0,$$

if defined for some $u<v\leqq m$. Let

$$q = (q_{ij0}), \qquad (i, j = u, u+1, \cdots, u+v). \tag{3}$$

Let F_1 be the field obtained by enlarging F to include the roots of the characteristic equation of q. We will from now on restrict ourselves to the field F_1 instead of F. For some suitably chosen matrix $t=(t_{ij0})$ we may reduce q to its classical canonical form

$$q' = t^{-1}qt = (q'_{ij0}).$$

Using

$$T = (T_{ij}(\gamma^2)\gamma^{\rho_{ij}}), \qquad (i, j = 1, 2, \cdots, m),$$

where

$$T_{ij}(\gamma^2) = t_{ij0}, \qquad (i, j = u, u+1, \cdots, u+v),$$

and otherwise

$$T_{ij}(\gamma^2) \text{ is zero for } i \neq j \text{ and 1 for } i = j,$$

we obtain

$$P = T^{-1}QT = (P_{ij}(\gamma^2)\gamma^{\rho_{ij}}),$$

where

$$P_{ij}(\gamma^2) = p_{ijw} + p_{ijw+2}\gamma^2 + \cdots,$$
$$(i, j = 1, \cdots, m;\ w = \rho_{ij}),$$
$$p_{ij0} = q'_{ij0}, \qquad (i, j = u, u+1, \cdots, u+v).$$

This argument may be repeated for any u for which (3) is true. All elements above the main diagonal are now congruent to zero modulo γ. Let $G(\lambda, \gamma^2)$ be the characteristic equation of P,

$$G(\lambda, \gamma^2) = |P - \lambda I| = g_0(\lambda) + g_2(\lambda)\gamma^2 + \cdots,$$

where

$$g_0(\lambda) = \prod_{i=1}^{m} (p_{ii0} - \lambda) = g_{10}(\lambda)g_{20}(\lambda)\cdots g_{s0}(\lambda),$$

$$g_{i0}(\lambda) = (\lambda_i - \lambda)^{r_i}, \qquad (\lambda_i \neq \lambda_j \text{ when } i \neq j),$$

that is, the λ_i $(i=1, 2, \cdots, s)$, represent the distinct values of p_{jj0}, $(j=1, 2, \cdots, m)$. We will denote by the set r_i those j for which $p_{jj0}=\lambda_i$.

$G(\lambda, \gamma^2)$ may be factored as follows.

$$G(\lambda, \gamma^2) = G_i(\lambda, \gamma^2)H_i(\lambda, \gamma^2), \qquad (i = 1, 2, \cdots, s),$$

where

$$G_i(\lambda, 0) = g_{i0}(\lambda), \qquad H_i(\lambda, 0) = h_{i0}(\lambda) = \prod_{j=1,\, j\neq i}^{s} g_{j0}(\lambda).$$

Such factorization may be made to depend upon the fact that the resultant* of $h_{i0}(\lambda)$ and $g_{i0}(\lambda)$ is not zero.

We now examine the matrix product

$$G_i(P, \gamma^2)H_i(P, \gamma^2) \equiv 0.$$

The columns of $H_i(P, \gamma^2)$ are vectors orthogonal to the matrix $G_i(P, \gamma^2)$. The elements in the main diagonal of $G_i(P, 0)$ are congruent modulo γ to

$$g_{i0}(\lambda_j)\begin{cases} = 0, & j \text{ in the set } \tau_i, \\ \not\equiv 0, & j \text{ not in the set } \tau_i; \end{cases}$$

above the main diagonal the elements are congruent to zero. The elements in the main diagonal of $H_i(P, 0)$ are congruent modulo γ to

$$h_{i0}(\lambda_j)\begin{cases} \not\equiv 0, & j \text{ in the set } \tau_i, \\ = 0, & j \text{ not in the set } \tau_i; \end{cases}$$

above the main diagonal they are congruent to zero.

Pick vectors $\xi_{i1}, \xi_{i2}, \cdots, \xi_{ir_i}$, $(i=1, 2, \cdots, s)$, as the columns numbered j of $H_i(P, \gamma^2)$, where j is in the set τ_i. This will determine $\sum_{j=1}^{s} r_j = m$ vectors ξ_{ij}. We propose to use ξ_{ij}, $(j=1, \cdots, r_i)$, as the τ_i, $(i=1, 2, \cdots, s)$, columns of the transforming matrix S. This insures that S is of admissible type. Further,

$$|S| \equiv \prod h_{i0}(\lambda_j) \not\equiv 0 \bmod \gamma^2, \qquad (i = 1, 2, \cdots, s; j \text{ in } \tau_i).$$

Therefore S^{-1} exists, and the vectors ξ_{ij}, $(i=1, \cdots, s;$ $j=1, \cdots, r_i)$, form a complete vector space.

It can be proved that the following lemma holds.

LEMMA. *Every vector η satisfying*

$$G_i(P, \gamma^2)\eta = 0,$$

where i is any one of the numbers $1, 2, \cdots, s$, is expressible as follows:

$$\eta = \sum_{j=1}^{r_i} a_j \xi_{ij}, \qquad (a_j \text{ are polynomials in } \gamma^2).$$

where the a_j are polynomials in γ^2.

* Bôcher, *Introduction to Higher Algebra*, pp. 195, 196.

We refer to the columns of S, picked and ordered as above, as vectors ξ_i, $(i=1, 2, \cdots, m)$.

Let η_i', $(i=1, \cdots, m)$, represent the rows of S^{-1}. Then the vectors η_i' and ξ_j form a biorthogonal system.

$$S^{-1}PS = \begin{Vmatrix} \eta_1' \\ \eta_2' \\ \cdot \\ \cdot \\ \eta_m' \end{Vmatrix} P(\xi_1, \xi_2, \cdots, \xi_m) = R = (R_{kj}),$$

where $R_{kj}=\eta_k P\xi_j$. If ξ_j is one of the set $(\xi_{i0}, \xi_{i1}, \cdots, \xi_{ir_i})$, then

$$G_i(P, \gamma^2)P\xi_j = PG_i(P, \gamma^2)\xi_j = 0,$$

and $P\xi_j$ is orthogonal to $G_i(P, \gamma^2)$. Hence, by the lemma above,

$$P\xi_j = \sum_{l=1}^{r_i} a_l\xi_i,$$

and $\eta_k P\xi_j=0$ for any k not one of the set τ_i.

Thus in R, any element R_{ij} corresponding to the intersection of P_{ii} and P_{jj}, where $p_{ii0}\neq p_{jj0}$, has been reduced to zero.

The complete solution of the problem may now be shown to rest upon the reduction to canonical forms of matrices M in which the elements along the main diagonal are congruent to each other modulo γ and those above the main diagonal are congruent to zero. If the elements $m_{ij}(\gamma^2)\gamma^{p_{ij}}$ of M are congruent to zero modulo $\gamma^{p_{ij}+2}$ for all i greater than j, we may repeat the preceding reduction using $\gamma^2(M-m_{110}I)$. Otherwise at this writing no general reduction has been obtained in this case. The problem, however, is still under consideration.

THE UNIVERSITY OF WISCONSIN

A GENERALIZATION OF HARMONIC FUNCTIONALS*

BY F. G. DRESSEL

1. *Introduction.* In a recent paper W. V. D. Hodge† showed that most of the elementary properties of harmonic functions could be extended to harmonic functionals; on the other hand, we can extend the notion of harmonic functionals so that most of these elementary properties persist in this larger class. The present paper presents such an extension, and properties of the functionals it contains.

2. *Generalized Harmonic Forms.* Consider the $(p-1)$-form

$$\phi = A_{i_1 \cdots i_{p-1}} dx^{i_1} \cdots dx^{i_{p-1}}, \quad (i = 1, \cdots, n), \tag{1}$$

in which the elements concerned obey the usual laws, with the exception that the dx's obey the non-commutative law of multiplication

$$dx^i dx^j = - dx^j dx^i. \tag{2}$$

Without loss of generality we assume that the summation in (1) is taken over all i for which $i_1 < \cdots < i_{p-1}$. If the A's have second partial derivatives which are continuous, then the form ϕ is said to be regular. The properties of such forms have been discussed by Cartan.‡

If the coefficients $A_{i_1 \cdots i_{p-1}}$ are symbols, we shall speak of (1) as a symbolic form. In the present paper we shall be concerned only with symbolic linear forms:

$$\alpha = \alpha_i dx^i, \ \beta = \beta_i dx^i, \ \cdots .$$

The rules of combination for these are the same as for the forms of the type (1), except that the commutative law for the multiplication of a symbolic and a non-symbolic form does not hold.

* Presented to the Society, February 23, 1935.

† W. V. D. Hodge, *A Dirichlet problem for harmonic functionals, with application to analytic varieties*, Proceedings of the London Mathematical Society, (2), vol. 36, part 4, pp. 257–303.

‡ E. Cartan, *Leçons sur les Invariants Intégraux*, 1922, Chapter 7.

In particular we note that if α_i, β_j are symbols and A is a function, the products $\alpha_i\beta_j=\beta_j\alpha_i$, $A\alpha_i$ are symbols; but $\alpha_i A$ is a function, since α_i has spent itself on the function A. Any form is said to be zero if all its coefficients are zero, and is indicated by writing $\phi=0$.

Let the multiplication of the form (1) by the symbolic linear form $\alpha=\alpha_i dx^i$ be indicated by ϕ_α,

$$\phi_\alpha = B_{i_1\cdots i_p} dx^{i_1}\cdots dx^{i_p},$$

$$B_{i_1\cdots i_p} = \sum_{k=1}^{p} (-1)^{k-1}\alpha_{i_k} A^{i_k}_{i_1\cdots i_p}, \quad (i_1 < \cdots < i_p), \tag{3}$$

where

$$A^{i_k}_{i_1\cdots i_p} = A_{i_1\cdots i_{k-1}i_{k+1}\cdots i_p}.$$

Denoting the adjoint of ϕ_α by ϕ^α, we have

$$\phi^\alpha = \sum \pm B_{i_1\cdots i_p} dx^{i_{p+1}}\cdots dx^{i_n},$$

the $+$ or $-$ sign being chosen according as $i_1, \cdots, i_n$ is an even or odd derangement of $1, \cdots, n$. Finally, let ϕ^α be multiplied by the symbolic form $\beta=\beta_i dx^i$, and let the result be indicated by

$$\Delta_\beta{}^\alpha\phi. \tag{4}$$

If γ is a cycle of $(p-1)$ dimensions, we shall define the integral $\int_\gamma\phi$ as an $\alpha\beta$-functional of γ if the form (4) is zero. Also in such a case, we shall speak of ϕ as an $\alpha\beta$-form.* We observe that if $\alpha_i=\beta_i=\partial/\partial x^i$, then ϕ_α is known in the literature as the covariant derivative of ϕ; and that if $\Delta_\beta{}^\alpha\phi=0$, and ϕ_α is regular, ϕ is a harmonic form and $\int_\gamma\phi$ a harmonic functional of γ.

3. *Properties of the $\alpha\beta$-forms.* We now give a series of theorems which are generalizations of the theorems of harmonic forms.

THEOREM 1. *The $B_{i_1}\cdots{}_{i_p}$ defined by* (3) *satisfy the relations*

$$\sum_{k=1}^{p+1} (-1)^{k-1}\alpha_{i_k} B^{i_k}_{i_1\cdots i_{p+1}} = 0. \tag{5}$$

* A more general functional could be obtained by taking α and β as any symbolic forms.

From the properties assigned to our symbolic forms, we see that the associative law holds, so that

$$\phi_{\alpha\alpha} = \alpha\alpha\phi = 0 \cdot \phi = 0,$$

where $\alpha\alpha = 0$ by virtue of (2); hence the theorem is immediate since the left side of (5) is a coefficient of $\phi_{\alpha\alpha}$.

THEOREM 2. *If $p = 1$ and ϕ is an $\alpha\beta$-form, then*

$$\beta_1\alpha_1\phi + \cdots + \beta_n\alpha_n\phi = 0. \tag{6}$$

The proof is easily supplied. We remark that any function satisfying the relation (6) will be called an $\alpha\beta$-function.

THEOREM 3. *If ϕ is an $\alpha\beta$-form, then the coefficients of ϕ_α are $\alpha\beta$-functions.*

Using the definition of ϕ^α, we have

$$\Delta_\beta{}^\alpha\phi = \sum \pm \left(\sum_{j=1}^{p} \beta_{i_j} B_{i_1 \cdots i_p} dx^{ij} \right) dx^{i_{p+1}} \cdots dx^{i_n} = 0, \tag{7}$$

and wish to show that

$$\alpha_j\beta_j B_{i_1 \cdots i_p} = 0 \tag{8}$$

for each set $i_1 < \cdots < i_p$.

We shall make use of the notation $B^{i_r}_{i_1 \cdots i_p, i_{p+v}}$ to mean that B in which i_r is missing from $i_1 < \cdots < i_p$ and that i_{p+v} is to be put into its natural order in $i_1 < \cdots < i_p$, a minus or a plus sign being taken according as i_{p+v} is taken over an odd or over an even number of i's. Note in (7), when finding the coefficient of $dx^{ir} dx^{ip+1} \cdots dx^{in}$, where $1 \leqq r \leqq p$, that besides the term $\beta_{i_r} B_{i_1 \cdots i_p}$, two types of terms can appear, those for which $i_{p+v} < i_r$ and those for which $i_{p+v} > i_r$. Taking this into account, we can write, except perhaps for a sign, this coefficient in the form

$$\beta_{i_r} B_{i_1 \cdots i_p} + (-1)^{p-r} \sum_{v=1}^{n-p} \beta_{i_{p+v}} B^{i_r}_{i_1 \cdots i_p, i_{p+v}}.$$

For an $\alpha\beta$-form the above symbol is zero, hence multiplying by α_{i_r} and summing with respect to r from 1 to p, we have

$$(9)\quad (-1)^p \sum_{r=1}^{p} \alpha_{i_r}\beta_{i_r}B_{i_1\cdots i_p} = \sum_{v=1}^{n-p}\beta_{i_{p+v}}\sum_{r=1}^{p}(-1)^{r-1}\alpha_{i_r}B^{i_r}_{i_1\cdots i_p,\, i_{p+v}},$$

where we have reversed the order of summation on the right hand side. If now $i_{k_v} < i_{p+v} < i_{k_v+1}$, where $k_v < p$, we can adjust the notation, so that Theorem 1 reads

$$\sum_{r=1}^{k_v}(-1)^{p+r-1}\alpha_{i_r}B^{i_r}_{i_1\cdots i_p,\, i_{p+v}} + \alpha_{i_{p+v}}B_{i_1\cdots i_p}$$

$$+\sum_{r=k_v+1}^{p}(-1)^{p+r-1}\alpha_{i_r}B^{i_r}_{i_1\cdots i_p,\, i_{p+v}} = 0.$$

Using this in (9), we get

$$\sum_{r=1}^{p}\alpha_{i_r}\beta_{i_r}B_{i_1\cdots i_p} = -\sum_{v=1}^{n-p}\beta_{i_{p+v}}\alpha_{i_{p+v}}B_{i_1\cdots i_p},$$

which leads immediately to the theorem.

4. *Forms in Two Sets of Differentials.* Turn now to the special case of $A_{i_1\cdots i_{p-1}}dx^{i_1}\cdots dx^{i_{p-1}}d\xi^{i_1}\cdots d\xi^{i_{p-1}}$, namely,*

$$U = A dx^{i_1}\cdots dx^{i_{p-1}}d\xi^{i_1}\cdots d\xi^{i_{p-1}}.$$

Let $\eta = \eta_i d\xi^i$ be a linear symbolic form and multiply U by it. Then

$$U_\eta = \left(\sum_{j=1}^{p}(-1)^{j-1}\eta_{i_j}A dx^{i_1}\cdots dx^{i_{j-1}}dx^{i_{j+1}}\cdots dx^{i_p}\right)d\xi^{i_1}\cdots d\xi^{i_p}.$$

The following theorem will be found useful.

THEOREM 4. *If $\beta_i A = \eta_i A$ (or if $\beta_i A = -\eta_i A$) and if A is an $\alpha\beta$-function, then U_η is an $\alpha\beta$-form.*

We may write $U_{\eta\alpha}$ in the form

$$U_{\eta\alpha} = \left(\sum_{j=1}^{p}\alpha_{i_j}\eta_{i_j}A dx^{i_1}\cdots dx^{i_p} + \sum_{k=p+1}^{n}\sum_{j=1}^{p}(-1)^{j-1}\right.$$

$$\left.\cdot\,\alpha_{i_k}\eta_{i_j}A dx^{i_k}dx^{i_1}\cdots dx^{i_{j-1}}dx^{i_{j+1}}\cdots dx^{i_p}\right)d\xi^{i_1}\cdots d\xi^{i_p}.$$

* The dx and $d\xi$ satisfy the law (2) separately, but in the present paper we assume that dx is commutative with $d\xi$. With this assumption U can be looked on as a symbolic form with $A d\xi^{i_1}\cdots d\xi^{i_{p-1}}$ as coefficients.

Forming the adjoint of the above with respect to the dx's, we have

$$(U_\eta)^\alpha = \sum \pm \Big(\sum_{j=1}^{p} \alpha_{i_j}\eta_{i_j} A dx^{i_{p+1}} \cdots dx^{i_n} + \sum_{k=p+1}^{n} \sum_{j=1}^{p} (-1)^{k+p} \cdot \alpha_{i_k}\eta_{i_j} A dx^{i_j} dx^{i_{p+1}} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n} \Big) d\xi^{i_1} \cdots d\xi^{i_p}.$$

Multiplying by β gives

$$\Delta_\beta{}^\alpha U_\eta = \sum \pm \Big(\sum_{v=1}^{p} \beta_{i_v} \Big[\sum_{j=1}^{p} \alpha_{i_j}\eta_{i_j} A \Big] dx^{i_v} dx^{i_{p+1}} \cdots dx^{i_n} + \sum_{k=p+1}^{n} \sum_{j=1}^{p} \beta_{i_k}\alpha_{i_k}\eta_{i_j} A dx^{i_j} dx^{i_{p+1}} \cdots dx^{i_n} + \sum_{v=1}^{p} \sum_{j=1}^{p} \sum_{k=p+1}^{n} (-1)^{k+p} \cdot \beta_{i_v}\eta_{i_j}\alpha_{i_k} A dx^{i_v} dx^{i_j} dx^{i_{p+1}} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n} \Big) d\xi^{i_1} \cdots d\xi^{i_p}.$$

If now $\beta_i A = \eta_i A$, we find that $\Delta_\beta{}^\alpha U_\eta$ is equal to

$$(10) \quad \sum \pm \Big(\sum_{v=1}^{p} \eta_{i_v} \Big[\sum_{j=1}^{n} \beta_{i_j}\alpha_{i_j} A \Big] dx^{i_v} dx^{i_{p+1}} \cdots dx^{i_n} \Big) d\xi^{i_1} \cdots d\xi^{i_p}.$$

The expression (10) has the opposite sign if $\beta_i A = -\eta_i A$. Thus if A satisfies the equation

$$\beta_1\alpha_1 A + \cdots + \beta_n\alpha_n A = 0,$$

U_η is an $\alpha\beta$-form.

If U^η is the adjoint of U_η with respect to the $d\xi$'s, we readily see from (10) that we have the following corollary.

COROLLARY. *If the conditions of Theorem* 4 *hold, then* U^η *is an* $\alpha\beta$*-form.*

4. *Special Cases.* In what follows we refer to a real euclidean space of n-dimensions, with $(x^1, \cdots, x^n)$ as a system of rectangular cartesian axes. The case treated by Hodge is obtained by specializing α and β to be

$$(11) \qquad \alpha = \beta = \frac{\partial}{\partial x^i} dx^i.$$

With this choice of α and β, Theorem 2 takes the form of Laplace's equation

$$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)\phi = 0, \qquad (x^i = x_i),$$

hence the name harmonic forms. With $\eta=(\partial/\partial\xi^i)d\xi^i$, the A of Theorem 4 can be taken

$$A = \frac{1}{r^{n-2}}, \text{ when } n > 2,$$

$$= \log r, \text{ when } n = 2,$$

where $r^2=\Sigma(x^i-\xi^i)^2$. If we take

$$\alpha = \frac{\partial}{\partial x^i}\,dx^i,$$

$$\beta = \frac{\partial}{\partial x^1}\,dx^1+\cdots+\frac{\partial}{\partial x^{n-1}}\,dx^{n-1}-dx^n, \tag{12}$$

$$\eta = \frac{\partial}{\partial \xi^1}\,d\xi^1+\cdots+\frac{\partial}{\partial \xi^{n-1}}\,d\xi^{n-1}-d\xi^n,$$

then Theorem 2 takes the form of the parabolic equation

$$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_{n-1}^2}-\frac{\partial}{\partial x_n}\right)\phi = 0.$$

Hence if α and β are defined by (12), we shall speak of an $\alpha\beta$-form as a parabolic form, and $\int_\gamma\phi$ as a parabolic functional. The A of Theorem 4 can be taken as

$$A = \frac{1}{(x_n-\xi_n)^{(n-1)/2}}\,e^{-[(x_1-\xi_1)^2+\cdots+(x_{n-1}-\xi_{n-1})^2]/[4(x_n-\xi_n)]}.$$

If α and β are defined by (11), with the exception that $\alpha_n=-\partial/\partial x^n$, then (6) of Theorem 2 takes the form of the hyperbolic equation, and we can refer to such $\alpha\beta$-forms as hyperbolic forms.

5. *Green's Theorem for Parabolic Functionals.* If ϕ and ψ are regular $(p-1)$-forms, Hodge has proved the following theorem of Green useful in the case of harmonic functionals:

THEOREM. *If α and β are given by* (11) *and D is any domain in our space bounded by the contour γ, we have*

$$\int_D (\Delta_\beta^{\alpha}\phi)\cdot\psi - \int_D (\Delta_\beta^{\alpha}\psi)\cdot\phi = \int_\gamma \phi^{\alpha}\cdot\psi - \int_\gamma \psi^{\alpha}\cdot\phi.^*$$

We shall now prove a corresponding theorem applicable to the case of parabolic functionals. Let α and β be defined by (12) and let

$$\bar{\beta} = \bar{\beta}_i dx^i = \frac{\partial}{\partial x^1} dx^1 + \cdots + \frac{\partial}{\partial x^{n-1}} dx^{n-1} + dx^n;$$

then our final theorem reads:

THEOREM 5. *If D is a region in our space bounded by the contour γ, and ϕ and ψ are regular $(p-1)$-forms, then*

$$\int_D (\Delta_\beta^{\alpha}\psi)\cdot\phi - \int_D (\Delta_\alpha^{\bar{\beta}}\phi)\cdot\psi = \int_\gamma [\psi^{\alpha}\cdot\phi]_n - \int_\gamma \phi^{\bar{\beta}}\cdot\psi,$$

where $[\psi^{\alpha}\cdot\phi]_n$ means that terms of $\psi^{\alpha}\cdot\phi$ not containing a dx^n are dropped.

Let

$$\psi = C_{i_1\cdots i_{p-1}} dx^{i_1}\cdots dx^{i_{p-1}},$$
$$\phi = A_{i_1\cdots i_{p-1}} dx^{i_1}\cdots dx^{i_{p-1}}.$$

Making use of (3) and (7), we have

$$(\Delta_\beta^{\alpha}\psi)\cdot\phi = (-1)^{n(p-1)} \cdot \sum \pm \left(\sum_{j=1}^{p}\sum_{k=1}^{p} (-1)^{j+k} A^{i_j}_{i_1\cdots i_p}\beta_{i_j}\alpha_{i_k} C^{i_k}_{i_1\cdots i_p}\right) dx^{i_1}\cdots dx^{i_n}.$$

Forming $(\Delta_\alpha^{\bar{\beta}}\phi)\cdot\psi$ and subtracting, we obtain

$$(\Delta_\beta^{\alpha}\psi)\phi - (\Delta_\alpha^{\bar{\beta}}\phi)\psi = (-1)^{n(p-1)}\sum \pm \left(\sum_{j=1}^{p}\sum_{k=1}^{p} (-1)^{j+k} \cdot [A^{i_j}_{i_1\cdots i_p}\beta_{i_j}\alpha_{i_k} C^{i_k}_{i_1\cdots i_p} - C^{i_k}_{i_1\cdots i_p}\alpha_{i_k}\bar{\beta}_{i_j} A^{i_j}_{i_1\cdots i_p}]\right) dx^{i_1}\cdots dx^{i_n}. \tag{13}$$

* For proof see W. V. D. Hodge, loc. cit., p. 266.

Two distinct cases arise:

Case 1. $\beta_{i_j} \neq \beta_n$,

$$A^{i_j}_{i_1 \cdots i_p} \beta_{i_j} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p} - C^{i_k}_{i_1 \cdots i_p} \alpha_{i_k} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}$$
$$= \beta_{i_j} (A^{i_j}_{i_1 \cdots i_p} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p}) - \alpha_{i_k} (C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}).$$

Case 2. $\beta_{i_j} = \beta_n$,

$$A^{i_j}_{i_1 \cdots i_p} \beta_{i_j} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p} - C^{i_k}_{i_1 \cdots i_p} \alpha_{i_k} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}$$
$$= - \alpha_{i_k} (C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p}).$$

A term from Case 1 multiplied by $dx^{i_1} \cdots dx^{i_n}$ and integrated leaves terms of the following type to be integrated over γ:

$$(-1)^{j-1} A^{i_j}_{i_1 \cdots i_p} \alpha_{i_k} C^{i_k}_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_{j-1}} dx^{i_{j+1}} \cdots dx^{i_n}$$
$$- (-1)^{k-1} C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n};$$

whereas from Case 2, we get on integration but the one term

$$- (-1)^{k-1} C^{i_k}_{i_1 \cdots i_p} \bar{\beta}_{i_j} A^{i_j}_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_{k-1}} dx^{i_{k+1}} \cdots dx^{i_n}.$$

Thus if we integrate (13) over D and make use of the above results, we see that the theorem follows.

It is interesting to note that for $n=2$, $p=1$, Theorem 5 gives

$$\int_D \left\{ \left[\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial \psi}{\partial x_2} \right] \phi - \left[\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial \phi}{\partial x_2} \right] \psi \right\} dx_1 dx_2$$
$$= \int_\gamma \phi \psi dx_1 + \left(\frac{\partial \psi}{\partial x_1} \phi - \frac{\partial \phi}{\partial x_1} \psi \right) dx_2.$$

This result is well known in the literature.*

DUKE UNIVERSITY

* Boundary value problems for the new functionals will be presented in a later article.

A NOTE ON THE EQUILIBRIUM POINT OF THE GREEN'S FUNCTION FOR AN ANNULUS

DEBORAH M. HICKEY

1. *Introduction.* In a previous paper* the motion of the equilibrium point of the Green's function for a plane annular region was studied as the pole was shifted along a radius in the neighborhood of the geometric mean circle C_0.† The expression for dr/dr_0 on C_0, r being the distance of the equilibrium point from the center of the circles, r_0 that of the pole, is $-F_{r_0}/F_r$, where

$$F_{r_0} = \frac{\partial F}{\partial r_0} = -\frac{2}{R}\left[\frac{1}{2\log R} - \frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m - 1}\right],$$

$$F_r = \frac{\partial F}{\partial r} = -\frac{2}{R}\left[\frac{1}{8} + \sum_{m=1}^{\infty}\frac{(-1)^m m}{R^m + 1}\right].$$

In these formulas 1 and R are the radii of the inner and outer circular boundaries of the region. It was shown by an application of a theorem of Schlömilch‡ that F_{r_0} does not vanish on C_0.

In this article this result and others are obtained by a method which seems better adapted to the problem.§

It is noticed that the function

$$f(z) = \frac{\pi}{\sin \pi z}\,\frac{z}{e^{az} - 1}, \qquad a = \log R,$$

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* D. M. Hickey, *The equilibrium point of Green's function for an annular region*, Annals of Mathematics, vol. 30 (1929), pp. 373–383.

† The Green's function for this region may be written in the form

$$g(M, M_0) = \log\frac{1}{MM_0} + \frac{1}{\log R}[\log R \log r_0 - \log r \log r_0/R]$$

$$- \sum_{m=1}^{\infty}\frac{1}{m}\,\frac{\cos m(\theta - \theta_0)}{R^{2m} - 1}\left\{r^m[r_0^m - r_0^{-m}] + r^{-m}\left[\left(\frac{R^2}{r_0}\right)^m - r_0^m\right]\right\}.$$

We take $F(r, r_0) = \partial g/\partial r$ for $r = r_0 = R^{1/2}$ and $\theta - \theta_0 = \pi$.

‡ *Über einige unendliche Reihen*, Zeitschrift für Mathematik und Physik, vol. 23 (1878), p. 132.

§ The suggestion that the method of contour integration and the theory of residues might prove useful was given by A. J. Maria.

where R is a real number greater than 1, has the sum of residues $\sum_{m=1}^{n}(-1)^m m/(R^m-1)$ within a suitably selected contour containing as singularities only the poles $1, 2, \cdots, n$ of $f(z)$. It turns out by an integration around a contour and certain limiting processes that an expression for F_{r_0} on C_0 is found which is a series of positive terms. The same method applied to a suitably chosen function $f(z)$ yields for the value of F_r a series of negative terms. These values show that dr/dr_0 is positive on C_0. By means of the preceding results it is proved that d^2r/dr_0^2 is negative on C_0.*

2. *The Evaluation of F_{r_0} and F_r on C_0 by Contour Integration.* The contour C_n chosen for the evaluation of F_{r_0} consists of the lines $x_n=n+1/2$, $y_n=\pm(2n+1)\pi/a$, semi-circular arcs that lie to the right of the imaginary axis of positive radius $\rho<\pi/(2a)$ and <1 and centers $\pm 2m\pi i/a$, $(m=0, 1, \cdots, n)$, and the straight line segments of the imaginary axis exterior to these arcs included between the upper and lower y_n lines. The function

$$f(z)=\frac{\pi}{\sin \pi z}\cdot\frac{z}{e^{az}-1}$$

is analytic inside and on C_n except at the poles $z=1, 2, \cdots, n$ of $\pi/\sin\pi z$. Hence the value of the integral $(1/2\pi i)\int_{C_n} f(z)dz$, where the contour C_n is traced in the counter-clockwise direction, gives $\sum_{m=1}^{n}(-1)^m m/(e^{am}-1)$, the sum of the residues of $f(z)$ inside C_n.

Let L_n be the straight line segments on the imaginary axis, K_n the semi-circular arcs, and S_n the remaining part of C_n. Over L_n the integral can easily be put into the form

$$-\frac{1}{2}\sum_{m=0}^{n-1}\int_{(2(m+1)\pi/a)-\rho}^{(2m\pi/a)+\rho}\frac{ydy}{\sinh \pi y}-\frac{1}{2}\int_{(2n+1)\pi/a}^{(2n\pi/a)+\rho}\frac{ydy}{\sinh \pi y}.$$

For the evaluation of the integral over the arc of K_n with center at $2m\pi i/a$, $(m\neq 0)$, a power series development of $f(z)$ about this point is used. Evaluated, the integral gives

$$-\frac{\pi^2}{a^2}\frac{m}{\sinh\,(2m\pi^2/a)}+P_m(\rho),$$

* It is evident that the corresponding results hold for any annulus.

where $P_m(\rho)$ is a power series in ρ with constant term zero. Over the arc with center at the origin the value is found to be $(-1/(2a))+P_0(\rho)$. Thus integration around C_n gives

$$\sum_{m=1}^{n}\frac{(-1)^m m}{e^{am}-1}=\frac{1}{2}\sum_{m=0}^{n-1}\int_{(2m\pi/a)+\rho}^{(2(m+1)\pi/a)-\rho}\frac{ydy}{\sinh \pi y}$$

$$+\frac{1}{2}\int_{(2n\pi/a)+\rho}^{((2n+1)\pi/a)-\rho}\frac{ydy}{\sinh \pi y}-\frac{1}{2a} \tag{1}$$

$$-\frac{2\pi^2}{a^2}\sum_{m=1}^{n}\frac{m}{\sinh (2m\pi^2/a)}+\sum_{m=-n}^{n}P_m(\rho)+\frac{1}{2\pi i}\int_{S_n}f(z)dz.$$

The value of $(1/2\pi i)\int_{C_n}f(z)dz$ is clearly the same for any positive ρ less than both $\pi/(2a)$ and 1. Letting ρ approach zero in (1), we obtain

$$\sum_{m=1}^{n}\frac{(-1)^m m}{e^{am}-1}=\frac{1}{2}\int_{0}^{(2n+1)\pi/a}\frac{ydy}{\sinh \pi y}$$

$$-\frac{2\pi^2}{a^2}\sum_{m=1}^{n}\frac{m}{\sinh (2m\pi^2/a)}-\frac{1}{2a}+\frac{1}{2\pi i}\int_{S_n}f(z)dz. \tag{2}$$

Now let n become infinite. The left member of (2) has as limit the convergent series $\sum_{m=1}^{\infty}(-1)^m m/(e^{am}-1)$. The first term on the right approaches the definite integral

$$\frac{1}{2}\int_{0}^{\infty}\frac{ydy}{\sinh \pi y},$$

which is known to have the value 1/8. The series approaches

$$-\frac{2\pi^2}{a^2}\sum_{m=1}^{\infty}\frac{m}{\sinh (2m\pi^2/a)}.$$

The integral over S_n has the limit zero.

To prove this last statement consider the modulus of $\int_{S_n}f(z)dz$. It can be shown* that over the entire curve S_n, $|1/\sin \pi z|$ and $|1/(e^{az}-1)|$ are bounded independently of n.

* That $|1/\sin \pi z|$ is bounded on S_n independently of n is proved essentially by Lindelöf in *Théorie des Résidus*, 1905, p. 32, footnote. The statement for $|1/(e^{az}-1)|$ can be proved in the same manner.

Let M be the greater of these two bounds. Moreover, on the upper and lower y_n lines

$$\left|\frac{1}{\sin \pi z}\right| < \frac{1}{\sinh((2_{n+1})\pi^2)/a)},$$

and on the right-hand boundary $z=(n+1/2)+iy$,

$$\left|\frac{1}{e^{az}-1}\right| < \frac{1}{e^{a(n+1/2)}-1}.$$

It then follows easily that

$$\left|\int_{S_n} f(z)dz\right| < \frac{Mk(2n+1)^2}{2a}\left[\frac{1}{2\sinh((2n+1)\pi^2/a)} + \frac{1}{e^{a(n+1/2)}-1}\right],$$

where k is a constant independent of n. This is sufficient to prove the statement.

In the limit for n infinite, (2) gives

$$(3)\quad \frac{1}{2a}-\frac{1}{8}+\sum_{m=1}^{\infty}\frac{(-1)^m m}{e^{am}-1} = -\frac{2\pi^2}{a^2}\sum_{m=1}^{\infty}\frac{m}{\sinh(2m\pi^2/a)}.$$

The left side of (3), where a is replaced by $\log R$, multiplied by $-2/R$, is F_{r_0}. Thus F_{r_0} is *positive*.

For the evaluation of F_r let

$$\frac{\pi}{\sin \pi z}\,\frac{z}{e^{az}+1}$$

be chosen for $f(z)$. Let the contour of integration C_n consist of the lines $x_n=n+1/2$, $y_n=\pm 2n\pi i/a$, semi-circular arcs to the right of the imaginary axis of radius $\rho<\pi/(2a)$ and with their centers at the points $\pm(2m+1)\pi i/a$, $(m=0, 1, \cdots, n)$, and the portions of the imaginary axis exterior to these arcs between the upper and lower y_n lines.

Applied to this function over the chosen contour, the method used above yields easily the result

$$(4)\quad \frac{1}{8}+\sum_{m=1}^{\infty}\frac{(-1)^m m}{e^{am}+1} = \frac{\pi^2}{a^2}\sum_{m=0}^{\infty}\frac{2m+1}{\sinh((2m+1)\pi^2/a)}.$$

By use of (4) with $\log R = a$, F_r can be written as

$$-\frac{2\pi^2}{R(\log R)^2}\sum_{m=1}^{\infty}\frac{(2m+1)}{\sinh\left(\dfrac{(2m+1)\pi^2}{\log R}\right)},$$

a series of *negative terms*. With these values for F_{r_0} and F_r, we conclude that $dr/dr_0 = -F_{r_0}/F_r$ on C_0 is positive.

3. *The Sign of* d^2r/dr_0^2 *on* C_0. From $dr/dr_0 = -F_{r_0}/F_r$, we calculate the second derivative

$$\frac{d^2r}{dr_0^2} = F_r^{-3}[2F_{r_0}F_rF_{r_0r} - F_{r_0}^2F_{rr} - F_r^2F_{r_0r_0}]. \tag{5}$$

From the general expressions for F_{r_0} and F_r in terms of r, r_0, and R, the following relations on C_0 are found to hold

$$F_{r_0r_0} = -R^{-1/2}F_{r_0}, \quad F_{rr} = -3R^{-1/2}F_r, \quad F_{r_0r} = -R^{-1/2}F_{r_0}.$$

A substitution of these values in (5) gives

$$\frac{d^2r}{dr_0^2} = R^{-1/2}F_{r_0}F_r^{-2}[F_{r_0} + F_r]. \tag{6}$$

Since F_{r_0} and F_r^2 are positive, the sign of d^2r/dr_0^2 on C_0 is that of $F_{r_0}+F_r$. From the results of the preceding section we have

$$F_{r_0} + F_r = \frac{2\pi^2}{R(\log R)^2}\sum_{m=1}^{\infty}\frac{(-1)^m m}{\sinh\left(\dfrac{m\pi^2}{\log R}\right)}.$$

This alternating series converges to a negative sum since its terms are in absolute value strictly decreasing to zero. This shows that d^2r/dr_0^2 is negative on C_0.

MISSISSIPPI DELTA STATE TEACHERS COLLEGE

NOTE ON A REMARK OF D. J. STRUIK ON CORRELATION COEFFICIENTS

BY Y. B. D. DERKSEN

In his paper on *Correlation and group theory*,* D. J. Struik points to the dualism between the total and partial correlation coefficients.

Let there be given any scatter diagram in n-dimensional space, consisting of N points determined by the coordinates

$$x_1^{(i)}, \quad x_2^{(i)}, \quad \cdots, \quad x_n^{(i)}, \qquad (i = 1, 2, \cdots, N).$$

For ease of computation suppose that

$$\bar{x}_j = \frac{1}{N} \sum_{i=1}^{N} x_j^{(i)} = 0, \qquad (j = 1, 2, \cdots, n).$$

Then, if

$$\sigma_{jk} = \frac{1}{N} \sum_{i=1}^{N} x_j^{(i)} \cdot x_k^{(i)},$$

the Gram determinant of the system is

$$G \equiv \begin{vmatrix} \sigma_{11} & \sigma_{12} \cdots \sigma_{1n} \\ \cdot & \cdot \\ \cdot & \cdot \\ \sigma_{n1} & \cdots \sigma_{nn} \end{vmatrix}.$$

Now let the minor of the element σ_{jk} be denoted by Σ_{jk}; then the equation of the so-called correlation hyperellipsoid may be written either in the form

$$\Sigma_{11}x_1^2 + 2\Sigma_{12}x_1x_2 + \cdots + \Sigma_{nn}x_n^2 = \text{const.}, \tag{1}$$

or, with tangential coordinates,

$$\sigma_{11}u_1^2 + 2\sigma_{12}u_1u_2 + \cdots + \sigma_{nn}u_n^2 = \text{const.} \tag{2}$$

The total correlation coefficients follow from

$$r_{jk} = \frac{\sigma_{jk}}{(\sigma_{jj} \cdot \sigma_{kk})^{1/2}}, \tag{3}$$

and the partial correlation coefficients are given by

* This Bulletin, vol. 36 (1930), pp. 869–878.

$$r_{jk\cdot 12\cdots(j)\cdots(k)\cdots n} = \frac{-\Sigma_{jk}}{(\Sigma_{jj}\Sigma_{kk})^{1/2}}. \tag{4}$$

The equations (1), (2), (3), and (4) show, according to Struik, that the total and partial correlation coefficients may be considered as the duals of each other.

We may ask if any statistical meaning can be attached to this view. This is answered by the following theorem.

Given any scatter diagram in n-dimensional space, it is possible to indicate a second, also in n-dimensional space, whose total correlation coefficients equal the opposites of the partial correlation coefficients of the first, and vice versa.

PROOF. Let the n coordinates of a point of a second scatter diagram be defined by

$$y_p^{(i_1,i_2,\cdots,i_{n-1})} = (-1)^p \begin{vmatrix} x_1^{(i_1)} & x_1^{(i_2)} & \cdots & x_1^{(i_{n-1})} \\ x_2^{(i_1)} & x_2^{(i_2)} & \cdots & x_2^{(i_{n-1})} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{p-1}^{(i_1)} & x_{p-1}^{(i_2)} & \cdots & x_{p-1}^{(i_{n-1})} \\ x_{p+1}^{(i_1)} & x_{p+1}^{(i_2)} & \cdots & x_{p+1}^{(i_{n-1})} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_n^{(i_1)} & x_n^{(i_2)} & \cdots & x_n^{(i_{n-1})} \end{vmatrix}, \tag{5}$$

$(p=1, 2, \cdots, n)$, where $(i_1, i_2, \cdots, i_{n-1})$ is any combination of $n-1$ numbers from the N integers.

If the total and partial correlation coefficients of this second scatter diagram are denoted by ρ_{jk} and $\rho_{jk\cdot 12\cdots(j)\cdots(k)\cdots n}$, then we shall prove

$$\rho_{jk} = -r_{jk\cdot 12\cdots(j)\cdots(k)\cdots n}, \tag{6}$$

$$\rho_{jk\cdot 12\cdots(j)\cdots(k)\cdots n} = -r_{jk}. \tag{7}$$

Consider now a well known theorem from the theory of determinants:

$$(8)\qquad \sum_{jk} = \frac{(-1)^{j+k}}{N^{n-1}}$$

$$\begin{vmatrix} \sum_1^N (x_1^{(i)})^2 & \sum_1^N x_1^{(i)} x_2^{(i)} & \cdots & \sum_1^N x_1^{(i)} x_{k-1}^{(i)} & \sum_1^N x_1^{(i)} x_{k+1}^{(i)} & \cdots & \sum_1^N x_1^{(i)} x_n^{(i)} \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \sum_1^N x_{j-1}^{(i)} x_1^{(i)} & \cdots & & & & & \\ \sum_1^N x_{j+1}^{(i)} x_1^{(i)} & \cdots & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \sum_1^N x_n^{(i)} x_1^{(i)} & \cdots & & & & & \end{vmatrix}$$

$$= \frac{1}{N^{n-1}} \sum_{i_1, i_2, \cdots, i_{n-1}} y_j^{(i_1, \cdots, i_{n-1})} y_k^{(i_1, \cdots, i_{n-1})},$$

$$(i_1 < i_2 < i_3 < \cdots < i_{n-1}).$$

Omitting the conditions $i_1 < i_2 < \cdots < i_{n-1}$, (8) becomes

$$9)\qquad \sum_{jk} = \frac{1}{N^{n-1}(n-1)!} \sum_{\substack{i_1, i_2, \cdots, i_{n-1} \\ (i_p \neq i_q)}} y_j^{(i_1, i_2, \cdots, i_{n-1})} y_k^{(i_1, \cdots, i_{n-1})},$$

for the number of permutations of a set of $n-1$ fixed values $i_1, i_2, \cdots, i_{n-1}$ is $(n-1)!$, and although the individual quantities $y_p^{(i_1, \cdots, i_{n-1})}$ may change sign by such a permutation, the product $y_j^{(i_1, \cdots, i_{n-1})} y_k^{(i_1, \cdots, i_{n-1})}$ never changes. The right member of (9) contains $N(N-1) \cdots (N-n+2)$ terms.

Now developing the determinant $y_p^{(i_1, \cdots, i_{n-1})}$ by the elements of the first column and summing with respect to i_1 only, we have $\sum_{i_1} y_p^{(i_1, \cdots, i_{n-1})} = 0$, as $\sum_{i_1=1}^N x_j^{(i_1)} = 0$, $(j = 1, 2, \cdots, n)$. Therefore $\sum_{i_1, \cdots, i_{n-1}} y_p^{(i_1, \cdots, i_{n-1})} = 0$. Let

$$s_{jk} = \frac{1}{N(N-1) \cdots (N-n+2)} \sum_{i_1, \cdots, i_{n-1}} y_j^{(i_1, \cdots, i_{n-1})} y_k^{(i_1, \cdots, i_{n-1})},$$

let G' denote the determinant obtained from G by replacing σ_{jk} by s_{jk}, and let S_{jk} be the minor of the element s_{jk} of G'. From the definition of the total correlation coefficient and (9), it follows that

$$(6)\qquad \rho_{jk} = \frac{s_{jk}}{(s_{jj}s_{kk})^{1/2}} = \frac{\Sigma_{jk}}{(\Sigma_{jj}\Sigma_{kk})^{1/2}} = -r_{jk\cdot 12\cdots(j)\cdots(k)\cdots n}.$$

From the formula for the partial correlation coefficient and (9) we get, using a theorem on reciprocal determinants,

$$(7)\qquad \rho_{jk\cdot 12\cdots(j)\cdots(k)\cdots n} = \frac{-S_{jk}}{(S_{jj}S_{kk})^{1/2}} = \frac{-\sigma_{jk}}{(\sigma_{jj}\sigma_{kk})^{1/2}} = -r_{jk}.$$

An interesting result can be obtained by a generalization of Struik's remark: *A partial correlation coefficient of the pth order may be considered as the dual of another partial correlation coefficient of the $(n-p-2)$th order.*

The statistical meaning of this statement is as follows.

If $r_{12\cdot(i+1)(i+2)\cdots n}$ is a partial correlation coefficient of the $(n-i)$th order of the first scatter diagram and $\rho_{12\cdot 3\cdots i}$ a like coefficient of the $(i-2)$th order of the system defined by (5), then making use of a theorem on reciprocal determinants,

$$\rho_{12\cdot 3\cdots i} = \frac{\begin{vmatrix} \Sigma_{12} & \cdots & \Sigma_{1i} \\ \Sigma_{32} & & \vdots \\ \vdots & & \vdots \\ \Sigma_{i2} & \cdots & \Sigma_{ii} \end{vmatrix}}{\begin{vmatrix} \Sigma_{22} & \cdots & \Sigma_{2i} \\ \vdots & & \vdots \\ \Sigma_{i2} & \cdots & \Sigma_{ii} \end{vmatrix}^{1/2} \cdot \begin{vmatrix} \Sigma_{11} & \Sigma_{13} & \cdots & \Sigma_{1i} \\ \Sigma_{31} & & & \vdots \\ \vdots & & & \vdots \\ \Sigma_{i1} & \cdots & \cdots & \Sigma_{ii} \end{vmatrix}^{1/2}}$$

$$= \frac{-\begin{vmatrix} \sigma_{21} & \sigma_{2\,i+1} & \cdots & \sigma_{2n} \\ \sigma_{i+1\,1} & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{n\,1} & \cdots & \cdots & \sigma_{nn} \end{vmatrix}}{\begin{vmatrix} \sigma_{11} & \sigma_{1\,i+1} & \cdots & \sigma_{1n} \\ \sigma_{i+1\,1} & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{n\,1} & \cdots & \cdots & \sigma_{nn} \end{vmatrix}^{1/2} \cdot \begin{vmatrix} \sigma_{22} & \sigma_{2\,i+1} & \cdots & \sigma_{2n} \\ \sigma_{i+1\,2} & & & \vdots \\ \vdots & & & \vdots \\ \sigma_{n2} & \cdots & \cdots & \sigma_{nn} \end{vmatrix}^{1/2}}.$$

Therefore

$$\rho_{12\cdot 3\cdots i} = -r_{12\cdot(i+1)(i+2)\cdots n}. \tag{10}$$

The numbering of the variables is immaterial; therefore generally,

$$\rho_{jk\cdot\{p\}} = -r_{jk\cdot\{n-p-2\}}, \tag{11}$$

where $\{p\}$ denotes any groups of p subcripts not including j or k, and $\{n-p-2\}$ the remaining ones. It is clear that (6) and (7) are particular cases of (11).

From the theory of partial correlation we know how to express partial correlations of order $n-2$ in terms of those of order $n-3$ and vice versa:

$$r_{12\cdot 34\cdots n} = \frac{r_{12\cdot 34\cdots(n-1)} - r_{1n\cdot 34\cdots(n-1)}\cdot r_{2n\cdot 34\cdots(n-1)}}{(1-r^2_{1n\cdot 34\cdots(n-1)})^{1/2}\cdot(1-r^2_{2n\cdot 34\cdots(n-1)})^{1/2}}, \tag{12}$$

$$r_{12\cdot 34\cdots(n-1)} = \frac{r_{12\cdot 34\cdots n} + r_{1n\cdot 23\cdots(n-1)}\cdot r_{2n\cdot 13\cdots(n-1)}}{(1-r^2_{1n\cdot 23\cdots(n-1)})^{1/2}\cdot(1-r^2_{2n\cdot 13\cdots(n-1)})^{1/2}}. \tag{13}$$

These formulas resemble the well known formulas

$$\rho_{12\cdot 3} = \frac{\rho_{12} - \rho_{13}\cdot\rho_{23}}{(1-\rho^2_{13})^{1/2}\cdot(1-\rho^2_{23})^{1/2}}, \tag{14}$$

$$\rho_{12} = \frac{\rho_{12\cdot 3} + \rho_{13\cdot 2}\cdot\rho_{23\cdot 1}}{(1-\rho^2_{13\cdot 2})^{1/2}\cdot(1-\rho^2_{23\cdot 1})^{1/2}}, \tag{15}$$

though (12) and (13) are usually derived in a rather complicated way (see Yule, *Theory of Statistics*, Chapter 12) that does not show this similarity. Equation (14) follows from (4) if we take $n=3$, $j=1$, $k=2$.

Now if we apply (10) (for $n=3$) to (14), we immediately get (15) and then replacing the subscript 3 in (14) and (15) by the subscript n and again applying (11), we obtain (12) and (13).

DEN HAAG, HOLLAND

NOTE ON THE GEOMETRIC INTERPRETATION OF THE VANISHING OF A CERTAIN PROJECTIVE INVARIANT OF TWO CONICS*

ROBIN ROBINSON

1. *Introduction.* The envelope of a line moving so that its pairs of intersections with two conics form a harmonic set is a third conic; the eight tangents to the given conics at their four points of intersection touch this envelope. The discriminant of this envelope is a relative invariant of weight four of the pair of ternary quadratic forms defining the given conic, and is the projective invariant mentioned in the title. When it vanishes, the envelope is degenerate, and the eight tangents mentioned above pass by fours through two points.†

Let the point-equations of the given conics be

$$\sum a_{ij}x_ix_j = 0, \quad a_{ij} = a_{ji}, \qquad \sum b_{ij}x_ix_j = 0, \quad b_{ij} = b_{ji}.$$

Then their line-equations are

$$\sum A_{ij}u_iu_j = 0, \qquad \sum B_{ij}u_iu_j = 0,$$

where A_{ij} is the cofactor of a_{ij} in the determinant $|a_{ij}|$, and similarly for B_{ij}. The line-equation of the envelope is then

$$\sum \alpha_{ij}u_iu_j = 0,$$

where α_{ij} is found by the Aronhold process:‡

$$\alpha_{ij} \equiv \sum \frac{\partial A_{ij}}{\partial a_{kl}} b_{kl} \equiv \sum a_{kl} \frac{\partial B_{ij}}{\partial b_{kl}}.$$

The condition that the envelope be degenerate is then $|\alpha_{ij}| = 0$. For want of a better term, we shall say that the two given conics are *related* if the envelope is degenerate. We shall proceed to prove a number of properties of related conics. Although most

* Presented to the Society, December 27, 1934.

† Salmon, *Conic Sections*, 1917, pp. 306 ff., 345; Clebsch-Lindemann, *Vorlesungen über Geometrie*, 1932, vol. 1, pt. 1, pp. 513, 523.

‡ Clebsch-Lindemann, loc. cit., pp. 501, 513.

of these properties are more general, we shall limit our discussion to related conics intersecting in four distinct points.

2. *Related Conics.* The locus of a point moving so that the lines joining it to the four points $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$, $A_4(1, 1, 1)$, respectively, have the cross-ratio λ is the conic

$$(1 - \lambda)x_1x_2 - x_2x_3 + \lambda x_3x_1 = 0.$$

Here λ represents not only the cross-ratio of the four lines, but also of the four points as points of the second order pencil lying on the conic. As λ varies we obtain a whole pencil of conics, the degenerate cases being given by $\lambda = 0, 1, \infty$. Let us determine under what conditions two non-degenerate conics of the pencil are related. The equation in line-coordinates of the conic C given by λ is readily found to be

$$u_1^2 + \lambda^2u_2^2 + (1 - \lambda)^2u_3^2 + 2\lambda u_1u_2 - 2\lambda(1 - \lambda)u_2u_3 + 2(1 - \lambda)u_3u_1 = 0.$$

If these coefficients are now polarized by the Aronhold process with respect to the coefficients of a second conic of the pencil, C', given by λ', the resulting coefficients α_{ij} are those for the envelope, whose equation is thus

$$u_1^2 + \lambda\lambda'u_2^2 + (1 - \lambda)(1 - \lambda')u_3^2 + (\lambda + \lambda')u_1u_2 + [2\lambda\lambda' - (\lambda + \lambda')]u_2u_3 + [2 - (\lambda + \lambda')]u_3u_1 = 0.$$

Its discriminant is

$$|\alpha_{ij}| = 4(\lambda + \lambda')[2 - (\lambda + \lambda')][2\lambda\lambda' - (\lambda + \lambda')],$$

and the condition that the two given conics be related is $|\alpha_{ij}| = 0$, that is, $\lambda + \lambda' = 0$, or $\lambda + \lambda' = 2$, or $1/\lambda + 1/\lambda' = 2$. These relations can also be written

$$\lambda' = -\lambda, \quad 1 - \lambda' = -(1 - \lambda), \quad 1 - 1/\lambda' = -(1 - 1/\lambda).$$

Since λ, $1 - \lambda$, and $1 - 1/\lambda$ and their reciprocals are the six cross-ratios of the four base-points in different orders, we may state the following theorem.

THEOREM 1. *A necessary and sufficient condition that two non-degenerate conics with four distinct intersections be related is that*

the cross-ratio of these four intersections in some order as points of one conic be the negative of the corresponding cross-ratio for the other conic.

The proper order of the intersections gives us a pairing of what we shall call *associated* intersections. For example, if $\lambda' = -\lambda$, since $\lambda = (A_1A_2, A_3A_4)$, A_1 and A_2 are associated intersections, as also are A_3 and A_4.

If the above three relations are written $\lambda' = -\lambda$, $\lambda' = 2 - \lambda$, $\lambda' = \lambda/(2\lambda - 1)$, the values $\lambda = -1, 2, 1/2$ are each seen to make one of the values of λ' correspond to a degenerate conic, and hence we have the following theorem.

THEOREM 2. *In a pencil of conics with four distinct base-points, there are just three non-degenerate conics which are related to a given non-degenerate conic, unless the base-points form a harmonic set on the given conic, when there are just two non-degenerate related conics. One of these related conics corresponds to each pairing of the four base-points, except when the two pairs separate one another harmonically.*

Let us now consider the tangents to two distinct non-degenerate conics $C(\lambda)$ and $C'(\lambda')$ at a common point, say $A_1(1, 0, 0)$. They are respectively

$$T_1: \quad (1 - \lambda)x_2 + \lambda x_3 = 0, \qquad T_1': \quad (1 - \lambda')x_2 + \lambda' x_3 = 0.$$

Solving simultaneously the equations of T_1 and

$$C': \quad (1 - \lambda')x_1x_2 - x_2x_3 + \lambda' x_3x_1 = 0,$$

we find that they intersect at A_1 and

$$S_1: \quad \left(1, \frac{\lambda' - \lambda}{1 - \lambda}, -\frac{\lambda' - \lambda}{\lambda}\right).$$

Similarly, T_1' meets C at A_1 and

$$S_1': \left(1, -\frac{\lambda' - \lambda}{1 - \lambda'}, \frac{\lambda' - \lambda}{\lambda'}\right).$$

For convenience we shall call S_1 and S_1' the *satellite points* of A_1.

Now S_1 and S_1' are collinear with $A_2(0, 1, 0)$ if and only if the determinant of the three sets of coordinates vanishes, that

is, if and only if $\lambda' = -\lambda$, in other words, if and only if C and C' are related conics with A_1 and A_2 associated intersections.

Similarly, S_1 and S_1' are collinear with $A_3(0, 0, 1)$ if and only if $\lambda' = 2-\lambda$, and with $A_4(1, 1, 1)$ if and only if $\lambda' = \lambda/(2\lambda-1)$.

It is clear from this that if $\lambda' = -\lambda$, the satellite points of A_2 are collinear with A_1, those of A_3 with A_4, and those of A_4 with A_3. Hence we may state the following theorem.

THEOREM 3. *A necessary and sufficient condition that two non-degenerate conics with four distinct intersections be related is that the satellite points of a point of intersection be collinear with a second point of intersection; when this is true, the satellite points of each intersection are collinear with the associated intersection.*

It is interesting to note that if the two conics are circles, then a necessary and sufficient condition that they be related with their finite intersections associated is that they be orthogonal.* This property suggests related conics as a natural projective generalization of orthogonal circles, which indeed they are.

That related conics are not a natural generalization of orthogonal conics in general is readily illustrated by the case of the ellipse $b_1^2x^2+a_1^2y^2=a_1^2b_1^2$ and the hyperbola $b_2^2x^2-a_2^2y^2=a_2^2b_2^2$, which are orthogonal if and only if confocal. The writer has found that they are related with their first and second quadrant intersections associated if and only if $b_1=b_2$.

Since in the case of related conics the envelope mentioned in the first paragraph is degenerate, the eight tangents T_i, T_i', $(i=1, 2, 3, 4)$, to C and C' at A_i, $(i=1, 2, 3, 4)$, pass by fours through two points. In case $\lambda' = -\lambda$, T_1, T_2, T_3', T_4' are concurrent at the point $(1, -\lambda, 1-\lambda)$, while T_3, T_4, T_1', T_2' are concurrent at $(1, \lambda, 1+\lambda)$. The two sets are perspective in the line $x_1-x_2+x_3=0$ (and also in another order in the line $x_1-x_2-x_3=0$), and hence have the same cross-ratio. From the equations

$$T_1: \quad (1-\lambda)x_2 + \lambda x_3 = 0, \qquad T_2: \quad (1-\lambda)x_1 - x_3 = 0,$$

$$T_3': \quad \lambda x_1 + x_2 = 0, \qquad T_4': \quad x_1 + \lambda x_2 - (1+\lambda)x_3 = 0,$$

we see that $T_3' \sim T_1+\lambda T_2$, $T_4' \sim \lambda T_1+T_2$, so that the cross-ratio is λ^2.

* Salmon, loc. cit., pp. 348 ff.

THEOREM 4. *If two conics are related, the eight tangents to them at their four points of intersection are concurrent in two perspective sets of four each, one of whose common cross-ratios is the square of the corresponding cross-ratio of the four corresponding intersections regarded as points of one of the conics.*

In case the two conics C and C' are related with A_1 and A_2 as associated intersections, $\lambda' = -\lambda$, and the satellite points of A_1 become

$$S_1: \left(1, \frac{-2\lambda}{1-\lambda}, 2\right), \qquad S_1': \left(1, \frac{2\lambda}{1+\lambda}, 2\right),$$

and their join becomes the line $x_3 = 2x_1$. Since this is independent of λ, we have the following theorem.

THEOREM 5. *If each non-degenerate conic of a pencil with four distinct base-points is paired with the related conic for which two definite base-points are associated intersections, the satellite points of one intersection all lie on the same straight line through the other intersection.*

3. *Related Line Conics.* It is of course desirable to consider the dual of two related conics. The locus of a point moving so that the four tangents drawn from it to two given conics form a harmonic set is a third conic, and the two given conics are related in the dual sense if this locus is degenerate. Suffice it to say that for the conics C and C' this locus has the equation

$$\begin{aligned}(\lambda - \lambda')^2(x_1^2 + x_2^2 + x_3^2) + 2[2 - (\lambda + \lambda')][2\lambda\lambda' - (\lambda + \lambda')]x_1x_2 \\ + 2(\lambda + \lambda')[2 - (\lambda + \lambda')]x_2x_3 \\ + 2(\lambda + \lambda')[2\lambda\lambda' - (\lambda + \lambda')]x_3x_1 = 0,\end{aligned}$$

and that its discriminant is

$$-16\lambda\lambda'(1-\lambda)(1-\lambda')(\lambda+\lambda')[2-(\lambda+\lambda')][2\lambda\lambda'-(\lambda+\lambda')].$$

Since neither C nor C' is degenerate, $\lambda\lambda'(1-\lambda)(1-\lambda') \neq 0$, and hence this locus is degenerate when and only when C and C' are related in the original sense. Hence we have the following theorems.

THEOREM 6. *Relatedness is a self-dual property.*

THEOREM 7. *Two non-degenerate conics with four distinct com-*

mon tangents are related if and only if the cross-ratio of these four common tangents in some order as lines of one conic is the negative of the corresponding cross-ratio for the other conic.

If the other tangents to C' and C drawn from the points of contact of a common tangent with C and C', respectively, be termed *satellite lines* of the common tangent, then we have the following theorem.

THEOREM 8. *The satellite lines of a common tangent to two non-degenerate conics are concurrent with another common tangent if and only if the conics are related.*

COROLLARY. *The satellite lines of a real common tangent to two orthogonal circles are concurrent with the other real common tangent.*

4. *Osculating Cubic of Related Conics.* In order to study the next property of related conics we shall introduce a non-homogeneous coordinate system in which A_1 is the origin, A_1S_1 and A_1S_1' the x and y axes, respectively, and S_1S_1' the line at infinity. Since then C is tangent to the x axis at the origin and passes through the point at infinity on the y axis, its equation is

$$C: \quad x^2 - axy - by = 0, \quad \text{or} \quad y = x^2/(ax + b).$$

Similarly,

$$C': \ -cxy + y^2 - dx = 0, \quad \text{or} \quad x = y^2/(cx + d).$$

Any cubic having a node at the origin with the axes as nodal tangents will have an equation of the form

$$Q_1: \ \alpha x^3 + \beta x^2y + \gamma xy^2 + \delta y^3 - bdxy = 0.$$

It is clear that the undetermined coefficients $\alpha, \beta, \gamma, \delta$ leave us just enough freedom to make Q_1 have four-point contact with each of C and C' at A_1. For example, if Q_1 and C are to have four-point contact at A_1, they must intersect five times there (four times for one branch of Q_1, once for the other). That is, if we substitute $y=x^2/(ax+b)$ in the left member of the equation of Q_1, x^5 must be a factor of the resulting expression. Substituting and multiplying by $(ax+b)^3$, we obtain the equation

$$\alpha x^3(ax + b)^3 + \beta x^4(ax + b)^2 + \gamma x^5(ax + b) + \delta x^6 - bdx^3(ax + b)^2 = 0.$$

Equating to zero the coefficients of x^3 and x^4, we have

$$\alpha b^3 - b^3 d = 0, \qquad \alpha = d,$$
$$3\alpha a b^2 + \beta b^2 - 2ab^2 d = 0, \qquad \beta = -ad.$$

Performing the same process with Q_1 and C', we obtain the relations $\gamma = -bc$, $\delta = b$. Hence the equation of Q_1 becomes

$$dx^3 - adx^2 y - bcxy^2 + by^3 - bdxy = 0.$$

This we shall call the *osculating cubic* of C and C' for the intersection A_1. Since Q_1 cuts each of C and C' five times at A_1, it must have a single intersection with each somewhere else. If we multiply the first equation given above for C by dx we obtain $dx^3 - adx^2 y - bdxy = 0$. Subtracting from the equation of Q_1, we have $-bcxy^2 + by^3 = 0$, and so the remaining intersection lies on the line $y = cx$, which also passes through the intersection of C' with the line at infinity other than S_1. In a similar manner, the remaining intersection of Q_1 and C' lies on $x = ay$, which also passes through the intersection of C with the line at infinity other than S_1'.

Now a necessary and sufficient condition that C and C' be related is that these extra intersections with the line at infinity be coincident, that is, $ac = 1$, in which case they also coincide with the extra points common to Q_1 and C, and to Q_1 and C'; and conversely. Hence we have the following theorem.

THEOREM 9. *A necessary and sufficient condition that two non-degenerate conics be related is that the osculating cubic at any intersection pass through another intersection, which in this case is the associated intersection.*

Curiously enough, it was this property in the case of two orthogonal circles which introduced the writer to this problem.

Returning now to our original coordinate system, and to the related conics C and C' for which $\lambda' = -\lambda$, I remark that it is merely a matter of detail which I shall spare the reader to show that the equation of the osculating cubic Q_1 at A_1 is

$$Q_1: \quad (3x_3 - 2x_1)x_2^2 + \lambda^2(x_3 + 2x_1)(x_2 - x_3)^2 = 0.$$

As λ changes, the cubic Q_1 for the pairs of conics C and C' form a pencil, with base-points as follows:

Quadruple base-point: $A_1(1, 0, 0)$, common node;
Double base-points: $(1, 0, -2)$, common tangent $x_3+2x_1=0$;
$(3, 2, 2)$, common tangent $3x_3-2x_1=0$;
Simple base-point: $A_2(0, 1, 0)$, associated intersection.

It is interesting to note that the common tangents at both double base-points pass through A_2. Moreover, $x_3+2x_1=0$ is the harmonic conjugate of S_1S_1' $(x_3=2x_1)$ with respect to the lines A_1A_2 $(x_3=0)$ and A_2A_3 $(x_1=0)$, while $3x_3-2x_1=0$ is the harmonic conjugate of S_1S_1' $(x_3=2x_1)$ with respect to A_1A_2 $(x_3=0)$ and A_2A_4 $(x_3=x_1)$.

By applying the involution

$$\rho x_1' = x_3 - x_2, \qquad \rho x_2' = x_3 - x_1, \qquad \rho x_3' = x_3,$$

which interchanges A_1 and A_2, A_3 and A_4, and leaves C and C' invariant, Q_1 is transformed into Q_2, the osculating cubic at A_2, which also passes through the associated intersection A_1:

$$Q_2: \quad (x_3 + 2x_2)(x_1 - x_3)^2 + \lambda^2(3x_3 - 2x_2)x_1^2 = 0.$$

Since every point of the line $x_3=x_1+x_2$ is invariant under the involution, three of the intersections of Q_1 and Q_2 lie on it. Two more are at each of A_1 and A_2. The only fixed point of the involution not on $x_3=x_1+x_2$ is the point (1, 1, 0), which lies on neither cubic, so the two remaining intersections of Q_1 and Q_2 are interchanged by the involution, and hence lie on a fixed line passing through (1, 1, 0). Suffice it to say that if λ is real they may be shown to be conjugate imaginary.

Similarly, the cubics Q_3 and Q_4, osculating cubics at A_3 and A_4, may be obtained. Three of the intersections of Q_1 or Q_2 with Q_3 or Q_4 lie on a line, while the other six lie on a conic.

Just one more property seems worthy of mention. The points S_1 and S_1' on the line $x_3=2x_1$ may be given by the relation

$$[(1 - \lambda)x_2 + 2\lambda x_1][(1 + \lambda)x_2 - 2\lambda x_1] = 0,$$

or $x_2^2-\lambda^2(x_2-2x_1)^2=0$. This same line meets Q_1 at A_2 and at the points given by $x_2^2+\lambda^2(x_2-2x_1)^2=0$. These two pairs of points separate one another harmonically, though the second pair is imaginary when λ is real.

DARTMOUTH COLLEGE

A GROUP OF OPERATIONS ON A PARTIALLY COLORED MAP

BY IRVING KITTELL

It has been previously shown* that if there be a minimum uncolorable map, and it is colored with the exception of one five-sided region in the four colors A, B, C, and D, then only one of the colors (say B) will be repeated in the five regions touching the region not colored, and that this ring of regions can be assigned the colors $DBABC$ in order. It has likewise been shown that when the map is thus colored, there will be two intersecting Kempe chains running from A to C and A to D, respectively. We shall say that any map, whether minimum uncolorable or not, which is partially colored in the above manner is colored *impasse*. The situation may be represented as in Fig. 1.

FIG. 1

The diagram does not attempt to show where or how many times the circuits cross one another. It merely means that they do cross. Here we have placed the uncolored five-sided region outside the rest of the map. We do this for compactness, but it must be noted that what follows is not dependent upon this choice of representation, nor upon the choice of lettering. Also to be remembered is the fact that right and left have only arbitrary meanings in discussing a map which may be deformed in any continuous manner over the surface of a sphere. Most writers would draw such a map with the uncolored region inside. We here call the D region the one in the diagram above on the right of A, and the C region the one on the left of A. We make the following definitions.

The region A is called the *vertex* of the ring. It is the region in this ring which lies between the two regions having the same color. We call the AC circuit the *left-hand circuit* and the AD circuit the *right-hand circuit*. We call the BD chain which includes the B region in the ring between A and C the *left-hand*

* P. J. Heawood, *Map colour theorem*, Quarterly Journal of Mathematics, vol. 24 (1890), pp. 332–338.

chain and the BC chain which includes the other B the *right-hand chain*. We call the CD chain which includes the C and D regions of the ring the *end tangent chain*; the BC chain touching the ring in the adjacent B and C regions the *left-hand tangent chain*; and the BD chain touching the ring in the adjacent B and D regions the *right-hand tangent chain*. Finally, we call the AB chain which includes the vertex, the *osculating chain*.

We now distinguish nine operations that can be performed on the map above.

α: transposing the colors of the left-hand chain.
β: transposing the colors of the right-hand chain.
γ: transposing the colors of the left-hand circuit.
δ: transposing the colors of the right-hand circuit.
ϵ: transposing the colors of the end tangent chain.
ζ: transposing the colors of the left-hand tangent chain.
η: transposing the colors of the right-hand tangent chain.
θ: transposing the colors of the osculating chain.
ι: leaving the coloring unchanged.

In all of these operations no color at all will be assigned to the outside region.

Now, unless each of these operations or any combination of these operations leaves the map in a form analogous to the first form above (that is, with two intersecting circuits running from the region between the two of the same color) it will be possible to color the map. Hence we may say that all of these operations are possible on a minimum uncolorable map. But it does not follow that any map on which all of these operations are possible is necessarily uncolorable.

Errera* describes explicitly only the operation α (or β) of these nine. Here we wish to study as exhaustively as possible all of the operations, and to see if there is any contradiction in their all being possible.

These nine operations are the generators of a group, which we will call the *impasse* group, and which consists of all possible combinations of these operations. In operating upon a map by two transformations in succession, it is to be noted that the first operation may move the vertex of the ring. Then any second

* Alfred Errera, *Du Coloriage des Cartes et de Quelques Questions d'Analysis Situs*, Thesis, Gauthier-Villars, Paris, 1921, 66 pp.

operation is referred to this vertex, for example, α transposes the chain on the left of this new vertex, etc. Thus, the operation α will move the vertex 144°, and hence, if it is performed twice, it will move the vertex through 288°, etc. We may note the following identities:

$$\iota^2 \equiv \gamma^2 \equiv \delta^2 \equiv \epsilon^2 \equiv \theta^2 \equiv \iota, \quad \alpha\beta \equiv \beta\alpha \equiv \iota, \quad \eta\zeta \equiv \zeta\eta \equiv \iota.$$

It follows that each of the generators has an inverse, and therefore each member of the impasse group has an inverse. Since we are dealing with a map of a finite number of regions, it is apparent that each of the operations of the group is periodic. The operation α has a period of at least twenty, since it can give twenty different arrangements of the colors of the five regions touching the uncolored region. It should be noted that in some maps α may have a period of forty, sixty, etc. The operation β also has an apparent period of twenty, while ζ and η have periods of fifteen. The operations γ, δ, ϵ, θ all have periods of precisely two.

We do not know how many members of the impasse group of an uncolorable map there may be, but we can show that there must be at least one hundred twenty. There are that many permutations of the colors of the five regions touching the uncolored region, and it can be shown that each one of these permutations can be effected by some operation of the impasse group. We shall not enumerate the operations for the various permutations, but merely assure the reader that such operations have been found. It can be shown that all of the permutations except twelve can be obtained by raising either the primitive operations or operations formed of pairs of the generators to various powers. For these twelve, we can use powers of triples. In all we can use powers of the following operations only: α, ζ, $\alpha\delta$, $\gamma\alpha$, $\alpha\gamma$, $\zeta\alpha$, $\alpha\zeta$, $\gamma\eta$, $\delta\eta$, $\epsilon\eta$, $\epsilon\beta\delta$, $\theta\zeta\gamma$, $\beta^2\epsilon$.

While we thus know that there must be at least 120 members of the impasse group, it must be remembered that there may in some cases be many more. For example, $(\beta\delta)^8$ and $(\alpha\gamma)^7$ each gives $DADBC$ for the ring, but the remainder of the map might conceivably be colored differently by these operations.

Errera presented a map on which the operation α could be repeated twenty times to return to the original coloring. At each

step the coloring remained impasse. We wish to point out a few additional facts about Errera's map, which is shown in Fig. 2. The operations α and ζ give the following results shown in Figs. 3 and 4.

FIG. 2

FIG. 3

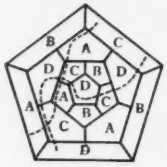

FIG. 4

We say that these operations are in this particular map equivalent, since one coloring can be obtained from the other by simply permuting $ABCD$ to $ADCB$ throughout the whole map. Repetitions of ζ give the diagrams shown in Figs. 5–7.

FIG. 5

FIG. 6

FIG. 7

Now note that ζ^4 is equivalent to ι, since this map could have been obtained from the original map by rotating through 144° and replacing $ABCD$ by $CBDA$. It is apparent, therefore, that ζ raised to any power is equivalent to ι, ζ, ζ^2, or ζ^3.

Similarly it can be shown that α^2 is not essentially different from ζ^2, nor α^3 from ζ^3 nor α^4 from ζ^4 or ι. Hence any power of α gives an impasse coloring. It can also be seen that $\alpha^{20} \equiv \iota$, and this is what Errera showed.

It can also be shown that θ, γ, δ, β, and η are equivalent to ζ^2, ι, ι, ζ^3, and ζ^3, respectively; that $\gamma\zeta$, $\delta\zeta$, $\beta\zeta$, $\eta\zeta$, and $\epsilon\zeta$ are equivalent to ζ, ζ, ι, ι, and ζ, respectively; that $\theta\zeta^2$, $\gamma\zeta^2$, $\delta\zeta^2$, $\beta\zeta^2$, and $\eta\zeta^2$ are equivalent to ι, ζ^2, ζ^2, ζ, and ζ, respectively; and that $\gamma\zeta^3$, $\delta\zeta^3$, $\beta\zeta^3$, $\eta\zeta^3$, and $\epsilon\zeta^3$ are equivalent to ζ^3, ζ^3, ζ^2, ζ^2, and ζ^3, respectively.

However, the operation ϵ gives us the diagram shown in Fig. 8. But this coloring is not impasse, since the circuits do not intersect. The operation $\theta\zeta$ gives the diagram shown in Fig. 9;

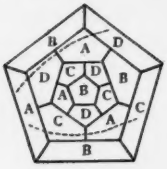

FIG. 8

FIG. 9

and this also fails to be impasse since the circuits are both broken. It may be easily shown that $\theta\zeta^2$ is likewise ruled out, as is $\epsilon\zeta^2$.

In conclusion, we say that the operations $\alpha, \beta, \gamma, \delta, \zeta, \eta$, are possible in any combinations or raised to any power, etc.; that these give rise to only three essentially different colorings (if we note that ζ^3 is merely a reflection of ζ); and that ϵ is possible when θ is impossible and only then, the former being possible on two of the three different colorings but not on the third.

Thus it is shown that there is no possibility of proving any contradiction between $\alpha, \beta, \gamma, \delta, \zeta$, and η, no matter in what combinations they are used, because in Errera's map we have a case in which they are not contradictory. However, the foregoing suggests that there may be a contradiction between ϵ, θ, and the other operations. If we can find some operation of the

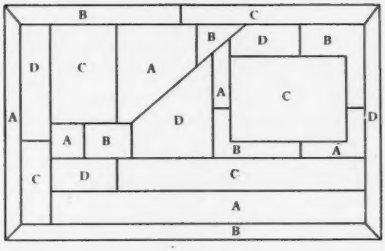

FIG. 10

impasse group or some set of operations such that no map will remain impasse under these operations, then the four-color problem will be solved.

The map shown in Fig. 10 is such that each of the primitive operations (that is, α, β, γ, δ, ϵ, ζ, η, θ) leaves the coloring impasse. This shows that no contradiction can be proved for the primitive operations taken singly.

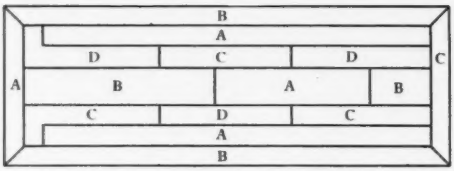

FIG. 11

However, in the map of Fig. 10, α^2 results in a coloring which is not impasse. Does this mean that there is a contradiction between ϵ and θ and α^2? The answer is negative, as is shown by the map of Fig. 11. Here ϵ and θ are each possible. Also α, α^2, α^3, and α^4 are each possible, but α^5 results in a non-impasse coloring.

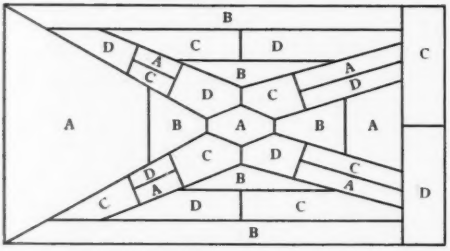

FIG. 12

We now give another map in Fig. 12, on which the operations γ, δ, ϵ are possible in any combinations. However, in this map none of the other operations (that is, α, β, ζ, η, θ) are possible.

But it shows at any rate that any effort to find a contradiction between γ, δ, ϵ, or any combination of these, would be futile.

We have failed to find a map on which all of the operations of the impasse group are possible.

NEW YORK UNIVERSITY

ON THE EQUIVALENCE OF TWO METHODS OF DEFINING STIELTJES INTEGRALS*

BY B. C. GETCHELL

1. *Introduction.* The Stieltjes integral, $\int_a^b \phi(x)dg(x)$, was originally defined for $\phi(x)$ continuous on the closed interval $[a, b]$, and $g(x)$ of bounded variation. The limit which gives rise to this integral is taken as the length of the greatest sub-interval approaches zero. The above restrictions on $\phi(x)$ and $g(x)$, however, are not at all necessary for the existence of the limit, although it fails when the two functions have a common point of discontinuity. A generalization which permits such discontinuities is obtained by taking the limit in the sense of subdivisions,† to be defined below. The Riemann integral is an instance of the first type of limiting process, while the associated Darboux integrals are of the subdivision type. These can be shown to be of the first type as well. It is the purpose of this note to obtain general conditions for the equivalence of the two limits. By the introduction of the notion of interval functions a simple restriction on the integrand is found to be both necessary and sufficient.

2. *Subdivisions.* By a *subdivision*, σ, of the linear interval $X=[a, b]$ will be understood a finite set of adjacent sub-intervals whose sum is X. The *norm* of σ, the length of the greatest sub-interval, will be written N_σ. By the *product*, $\sigma' \cdot \sigma''$, of two subdivisions of X will be understood the subdivision which consists of all products of the form $I' \cdot I''$, where I' is an element of σ', and I'' is an element of σ''. It is assumed that every such

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* Presented to the Society, April 6, 1934.

† Moore and Smith, *A general theory of limits*, American Journal of Mathematics, vol. 44 (1922), pp. 102–121.

pair of subdivisions has a product. Whenever two subdivisions σ and σ' of X satisfy the condition $\sigma = \sigma \cdot \sigma'$ we shall write $\sigma \geqq \sigma'$. This is equivalent to having every element of σ contained in or identical with some element of σ'.

3. *Interval Functions.* An *interval function*, $f(I)$, on X will be any relation which assigns one or more finite real values to each sub-interval I which is an element of a subdivision of X. The expression $f(I) = \phi(\xi)[g(x'') - g(x')]$, $x' \leqq \xi \leqq x''$, where $\phi(x)$ and $g(x)$ are bounded point functions, and x', x'' are the left and right end points of I, may be taken as an example.

4. *Integrals.* The symbol $\sum_\sigma f(I)$ will be used to designate any sum of the values of $f(I)$ obtained by choosing one value to correspond to each element of σ. It is a function of σ and may be multiple valued. The *norm* and *sigma* integrals of $f(I)$ on X, written $N\int_X f(I)$ and $\sigma\int_X f(I)$, respectively, will be the corresponding limits of $\sum_\sigma f(I)$ with respect to the subdivisions of X. These limits are defined explicitly as follows.

(1) *The norm limit.* If there exists a finite number A such that for every positive number e a positive N_e can be found which makes* $LUB \left| \sum_\sigma f(I) - A \right| \leqq e$ whenever σ is a subdivision of X with $N_\sigma \leqq N_e$, we write $\lim_N \sum_\sigma f(I) = A$.

(2) *The sigma limit.* If there exists a finite number A such that for every positive e there can be found a subdivision σ_e of X with respect to which $LUB \left| \sum_\sigma f(I) - A \right| \leqq e$ whenever $\sigma \geqq \sigma_e$, we write $\lim_\sigma \sum_\sigma f(I) = A$. This is the limit in the sense of subdivisions mentioned above. It is easily shown to exist and to be equal to the norm limit when the latter exists.

5. *Additive and Pseudo-Additive Functions.* Let I' and I'' be a pair of adjacent intervals with the common end point x. Consider now the expression

$$\Delta(I, x) = LUB \left| f(I') + f(I'') - f(I) \right|,$$

where $I = I' + I''$. The function $f(I)$ is said to be *additive* if $\Delta(I, x)$ is zero for every such pair. We shall need a weaker condition. If $\lim \Delta(I, x) = 0$ as the length of I goes to zero, we shall say that $f(I)$ is *pseudo-additive* at x. The function will be called pseudo-additive on the interval X if it has the property at every

* LUB = least upper bound.

point. When elements of zero length (points) are permitted, it becomes possible for three sub-intervals to be adjacent in the sense that they have a common end point. Then $\Delta(I, x)$ must take the form

$$LUB\,|\,f(I') + f(I'') + f(I''') - f(I)\,|\,.$$

The continuous interval functions are necessarily pseudo-additive.

6. *Condition for Equivalence of Integrals.* As stated in the introduction, the sigma integral may exist when the corresponding norm integral fails. The following theorem gives the condition for their equivalence.

THEOREM. *For the existence of $N\int_X f(I)$ it is necessary and sufficient that $\sigma\int_X f(I)$ exist and that $f(I)$ be pseudo-additive on X.*

Necessity. Since the existence of the norm integral implies that of the sigma integral, it must be shown only that $f(I)$ is pseudo-additive on X whenever the former integral exists. If the contrary is assumed, there must be a point x of X and a positive number e_0 such that for every $N>0$ there can be found a pair of adjacent intervals I' and I'' of X with the following properties: I' and I'' have x as a common end point, their sum, I, has a length not greater than N, and $LUB|f(I')+f(I'')-f(I)| > e_0$. Now choose a subdivision σ' of X with I' and I'' as elements and with norm less than or equal to N. Let σ'' be the subdivision obtained by the substitution of I for its components in σ'. Then

$$LUB\left|\sum_{\sigma'} f(I) - \sum_{\sigma''} f(I)\right| \geqq LUB|f(I')+f(I'')-f(I)| > e_0.$$

Since N is arbitrary, the hypothesis that $N\int_X f(I)$ exists cannot hold.

Sufficiency. From the definition of the sigma integral it follows that for every positive e there exists a subdivision σ_e of X such that

$$LUB\left|\sum_{\sigma} f(I) - \sigma\int_X f(I)\right| \leqq e/2 \tag{1}$$

whenever $\sigma \geqq \sigma_e$. Let $x_1, \cdots, x_n$ be the end points of the elements of σ_e. If k is the number of its sub-intervals, it is possible to find, because $f(I)$ is pseudo-additive, an $N_e' > 0$ such that

$$LUB\,|\,f(I') + f(I'') - f(I)\,| \leqq e/(2k) \tag{2}$$

whenever the length of I is not greater than N_e', and one of the points $x_1, \cdots, x_n$ is a common end point of I' and I''. If N_e'' is one-half the least distance between any two of the above points, let the lesser of N_e' and N_e'' be designated by N_e. Then, by reason of (2), it follows that

$$LUB\,|\,\textstyle\sum_{\sigma'} f(I) - \sum_{\sigma} f(I)\,| \leqq ke/(2k) = e/2 \tag{3}$$

whenever $N_{\sigma'} \leqq N_e$ and $\sigma = \sigma' \cdot \sigma_e \geqq \sigma_e$. Therefore

$$LUB\left|\,\textstyle\sum_{\sigma'} f(I) - \sigma \int_X f(I)\,\right| \leqq e$$

by reason of (1) and (3), and the norm integral exists. It is assumed that the points $x_1, \cdots, x_n$ are all distinct, that is, that sub-intervals of zero length are excluded. The proof is easily modified to allow for this case also.*

EXAMPLE 1. The elements of subdivisions in the first four examples will be closed intervals, and $f(I) = |\,g(x'') - g(x')\,|$, where x' and x'' are the left and right end points of I. If $g(x)$ is of bounded variation, then $\sigma \int_X f(I)$ exists and is equal to the total variation of $g(x)$ on X. The theorem just proved gives the well known condition

$$|\,g(x) - g(x-0)\,| + |\,g(x+0) - g(x)\,| = |\,g(x+0) - g(x-0)|$$

at each inner point of X for the existence of the total variation as a norm integral.

EXAMPLE 2. $f(I) = \phi(\xi)\,[g(x'') - g(x')]$, $x' \leqq \xi \leqq x''$; $\phi(x)$ and $g(x)$ are bounded point functions. In this case $\Delta(I, x)$ takes the form

$$LUB\,|\,[\phi(\xi') - \phi(\xi)]\,|g(x) - g(x')]$$
$$+ [\phi(\xi'') - \phi(\xi)]\,[g(x'') - g(x)]\,|,$$

* After this note had been sent to press, T. H. Hildebrandt called the author's attention to an article by Gustav Doetsch, Journal für Mathematik, August, 1934, in which the property referred to here as *pseudo-additive* is shown to be sufficient for the existence of the norm integral in the special case of monotonic interval functions. Application is made to extensions of the Darboux theorem on upper and lower integrals.

$x' \leqq \xi' \leqq x \leqq \xi'' \leqq x''$. Lim $\Delta = 0$ at any point where $\phi(x)$ or $g(x)$ is continuous. To show that the converse is also true we assume that they have a common point of discontinuity at x. Then there exists an $e_0 > 0$ such that for every positive number N two points x_1 and x_2 can be found which satisfy the inequalities $|x_2 - x| \leqq |x_1 - x| \leqq N/2$, $|g(x_1) - g(x)| > \sqrt{e_0}$, and $|\phi(x_2) - \phi(x)| > \sqrt{e_0}$. Without loss of generality it may be assumed that $x_1 < x$. In the above expression for Δ let $x' = x_1$ and $x'' = x + N/2$. If $x_2 < x$ let $\xi = \xi'' = x$ and $\xi' = x_2$. On the other hand, if $x < x_2$ take $\xi = \xi'' = x_2$ and $\xi' = x$. In either case

$$\Delta(I, x) \geqq |\,[\phi(x_2) - \phi(x)][g(x) - g(x_1)]\,| > (\sqrt{e_0})^2 = e_0.$$

But $|x'' - x'| \leqq N$, an arbitrary number. It follows that $f(I)$ is pseudo-additive on X if and only if $\phi(x)$ and $g(x)$ have no common point of discontinuity.

EXAMPLE 3. Substitute $x' < \xi < x''$ for $x' \leqq \xi \leqq x''$ in Example 2. The resulting integrals* are more general but reduce to the corresponding integrals of the earlier example whenever these exist. The condition that $f(I)$ be pseudo-additive is the same, with the exception that $\phi(x)$ and $g(x)$ may have removable discontinuities at the same point. If either discontinuity is not of this type, the norm integral fails to exist, unless the point in question is an end point of X. Consider the case $\phi(x) = g(x) = 0$, $x \neq x_0$; $\phi(x_0) = g(x_0) = 1$, x_0 an inner point of X. Both integrals exist and have the value zero. Under the definition of $f(I)$ in the previous example neither integral exists.

EXAMPLE 4. $f(I) = [\phi(x'') + \phi(x')][g(x'') - g(x')]/2$.†

$$2\Delta(I, x) = \begin{vmatrix} g(x) & \phi(x) & 1 \\ g(x') & \phi(x') & 1 \\ g(x'') & \phi(x'') & 1 \end{vmatrix}.$$

If the sigma integral exists, the vanishing of the determinant at each inner point of X as x' and x'' approach x independently from opposite sides is a necessary and sufficient condition for the existence of the norm integral.

* B. Dushnik, Dissertation, University of Michigan, 1931, p. 13.

† H. L. Smith, Transactions of this Society, vol. 27 (1925), p. 491. See also H. S. Kaltenborn, Dissertation, University of Michigan, 1934.

EXAMPLE 5. Subdivisions consist of open intervals and points. $f(I)=\phi(\xi)[g(x''-0)-g(x'+0)]$, $x'<\xi<x''$, if I is an interval. For the point x, $f(I)=\phi(x)[g(x+0)-g(x-0)]$.* It is assumed that both functions are bounded and that $g(x)$ has no discontinuities of the second kind. In view of the remark concerning subdivisions with point elements we obtain an expression for $\Delta(I, x)$ which may be written as the least upper bound of

$$\begin{aligned}&\big|[\phi(x)-\phi(\xi)][g(x+0)-g(x-0)]\\&\quad+[\phi(\xi')-\phi(\xi)][g(x-0)-g(x'+0)]\\&\quad+[\phi(\xi'')-\phi(\xi)][g(x''-0)-g(x+0)]\big|.\end{aligned}$$

The test in this case is the behavior of the first term; the remainder of the expression always goes to zero with the length of I. Therefore $\lim\Delta=0$ if and only if $\phi(x)$ is continuous or $g(x)$ has at most a removable discontinuity at x.

If $g(a)$ and $g(b)$ are substituted for $g(a-0)$ and $g(b+0)$ respectively in the above then $N\int_X f(I)$ reduces to the Young integral† $\int_a^b\phi(x)dg(x)$ obtained by integrating

$$\begin{aligned}F(I)=\phi(\xi)[g(x''-0)-g(x'+0)]+\phi(x')[g(x'+0)-g(x')]\\+\phi(x'')[g(x'')-g(x''-0)],\end{aligned}$$

$x'<\xi<x''$, over the closed interval $X=[a, b]$, with subdivisions consisting of closed intervals as in the previous examples. The conditions for the existence of the Young integral are the same as those at the end of the preceding paragraph.

NEEDHAM, MASS.

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* A. Kolmogoroff, *Untersuchungen über den Integralbegriff*, Mathematische Annalen, vol. 103 (1930), pp. 654–696.

† W. H. Young, *On integration with respect to a function of bounded variation*, Proceedings of the London Mathematical Society, (2), vol. 13 (1913), p. 113.

A NEW PROOF OF MINKOWSKI'S THEOREM ON THE PRODUCT OF TWO LINEAR FORMS

BY R. Q. SEALE

Remak,* Mordell,† Landau,‡ and Blichfeldt§ have all proved the theorem, first proved by Minkowski‖ (1901):

If $\alpha, \beta, \gamma, \delta, \xi_0, \eta_0$ are real and $\alpha\delta-\beta\gamma=1$, integers x, y always exist such that

$$|(\alpha x+\beta y-\xi_0)(\gamma x+\delta y-\eta_0)| \leqq \frac{1}{4}.$$

This theorem includes as a special case, the classical theorem of Tchebychef¶ (1866):

If a is irrational and b is real, an infinite number of pairs of integers x, y $(y>0)$ always exist such that $|(x-ay-b)|$ can be made arbitrarily small, and at the same time

$$|x-ay-b|<\frac{2}{y}.$$

In what follows, by making use of no principles more advanced than the elementary properties of convergents, I have proved three theorems, the first one being the Tchebychef theorem stated above. The second is Minkowski's theorem on the product of two homogeneous forms, while the third is the Minkowski theorem stated above. I feel that, although Tchebychef's theorem is a special case of Minkowski's theorem, its

* Bachmann, *Die Arithmetik der Quadratischen Formen*, Zweite Abteilung, p. 66; or Remak, Journal für die reine und angewandte Mathematik, vol. 142, p. 278.

† Mordell, Journal of the London Mathematical Society, vol. 3 (1928), p. 19.

‡ Landau, Journal für die reine und angewandte Mathematik, vol. 165 (1931), p. 1.

§ Published in a syllabus which Professor Blichfeldt distributed to a class in geometry of numbers at Stanford University, winter and spring quarters, 1932.

‖ Minkowski, *Diophantische Approximationen*, pp. 42–45.

¶ Œuvres de Tchebychef, vol. 1, p. 637.

proof should be included in this paper because it so completely illustrates the simplicity of the methods I have used.

Professor Blichfeldt, in his proof, shows that if the ratio α/β is irrational, the form $|(\alpha x+\beta y-\xi_0)|$ can be made arbitrarily small at the same time that Minkowski's theorem is satisfied. This is not shown in any of the other proofs I have read, although it follows from the method Minkowski used in his proof.

THEOREM 1. *Integers x, y $(y\neq 0)$ always exist such that $|x-\theta y-\omega|$ can be made arbitrarily small, and at the same time*

$$|x-\theta y-\omega|<\frac{1}{4|y|}, \tag{1}$$

where θ is irrational and ω is not an integer.

PROOF. We use the following lemmas.

LEMMA A. *If m and n are relatively prime integers, $nx-my=N$ has a solution in integers x_0, y_0, such that $|y_0|\leqq n/2$.*

LEMMA B. *If $e_1^2=e_2^2=e_3^2=1$, and $a\leqq 1$, $b\leqq 1$, $c\leqq 1$, at least one of the inequalities*

$$|a(e_1e_3ab-e_2c)|\leqq 1, \tag{a}$$

and

$$|(2-a)[e_1e_3(2-a)b+e_2c]|\leqq 1, \tag{b}$$

is true.

To prove this lemma, assume that

$$a(ab+c)>1. \tag{c}$$

This assumption is permissible, since (a) is surely true if (e_1e_3) and e_2 agree in algebraic sign.

Then if (b) is false we must have either

$$(2-a)[c-(2-a)b]>1, \tag{d}$$

or

$$(2-a)[c-(2-a)b]<-1. \tag{e}$$

From (c) we obtain $1\geqq b>(1-ac)/a^2$. If, however, we re-

place b by $(1-ac)/a^2$ in (d), the inequality is certainly not weakened. That is,

$$(2-a)\left[c-(2-a)\frac{1-ac}{a^2}\right]$$
$$=(2-a)\left(\frac{a^2c-2+2ac+a-a^2c}{a^2}\right)$$
$$=(2-a)\left(\frac{-2+2ac+a}{a^2}\right)>1$$

is true. But this reduces to

$$c[(a-1)^2-1]+1+(a-1)^2<0,$$

which is impossible. Hence, (c) and (d) are not both possible.

From (c) we obtain $1\geqq c>(1-a^2b)/a$, and if we replace c by the right-hand side of this inequality, (e) is not weakened. That is,

$$(2-a)\left[\frac{1-a^2b}{a}-(2-a)b\right]=(2-a)\left(\frac{1-a^2b-2ab+a^2b}{a}\right)$$
$$=(2-a)\left(\frac{1-2ab}{a}\right)<-1$$

is true. But this reduces to

$$1-2ab+a^2b=1+[(1-a)^2-1]b<0,$$

which is impossible. The lemma is therefore true.

Now let us define P by means of the equation

$$P=x-\theta y-\omega, \tag{2}$$

then express θ as a continued fraction, and let m/n be *any* convergent to θ. If we write

$$\frac{m}{n}-\theta=\frac{e_1b}{n^2}, \tag{3}$$

we have $b<1$ and $e_1^2=1$.

Define N and k by the equation

$$n\omega=N+k, \tag{4}$$

where N is an integer and $|k| \leqq 1/2$. Say that

$$k = \frac{e_2 c}{2}. \tag{5}$$

Then $e_2^2 = 1$, and $c \leqq 1$.

We can now write

$$\begin{aligned} P &= \frac{nx - my - N}{n} + \left(\frac{m}{n} - \theta\right) y - \frac{k}{n} \\ &= \frac{nx - my - N}{n} + Q. \end{aligned} \tag{6}$$

From (3), (5), and (6), we obtain

$$Q = y\left(\frac{m}{n} - \theta\right) - \frac{k}{n} = \frac{e_1 by}{n^2} - \frac{e_2 c}{2n}. \tag{7}$$

By Lemma A, $nx - my - N = 0$ has a solution (x_0, y_0), both integers, with $|y_0| \leqq n/2$. Since ω is not an integer, $y_0 \neq 0$. Say that

$$y_0 = \frac{e_3 an}{2}, \tag{8}$$

where $0 < a \leqq 1$, and $e_3^2 = 1$.

A second solution of $nx - my - N = 0$ will be $x_1 = x_0 - e_3 m$, $y_1 = y_0 - e_3 n$. Then

$$y_1 = y_0 - e_3 n = -\frac{e_3 n}{2}(2 - a). \tag{9}$$

When $x = x_0$, $y = y_0$, $Q = Q_0$, that is, from (7) and (8),

$$\begin{aligned} Q_0 &= y_0\left(\frac{m}{n} - \theta\right) - \frac{k}{n} = \frac{e_3 an}{2} \cdot \frac{be_1}{n^2} - \frac{e_2 c}{2n} \\ &= \frac{1}{2n}(e_1 e_3 ab - e_2 c). \end{aligned} \tag{10}$$

Similarly, for $x_1 = x_0 - e_3 m$, $y_1 = y_0 - e_3 n$, we obtain

$$Q_1 = (y_0 - e_3 n)\left(\frac{m}{n} - \theta\right) - \frac{k}{n} = -\frac{e_3 n}{2}(2-a)\frac{e_1 b}{n^2} - \frac{e_2 c}{2n}$$

(11)

$$= -\frac{1}{2n}[e_1 e_3(2-a)b + e_2 c].$$

Both (10) and (11) can be made arbitrarily small, numerically, for large enough n, and therefore $|x_0 - \theta y_0 - \omega|$ and $|x_1 - \theta y_1 - \omega|$ can both be made arbitrarily small for large enough n.

The corresponding values of $4y_0Q_0$ and $4y_1Q_1$ are

$$4y_0Q_0 = 4\frac{e_3 an}{2}\cdot\frac{1}{2n}(e_1 e_3 ab - e_2 c) = e_3 a(e_1 e_3 ab - e_2 c), \tag{12}$$

$$4y_1Q_1 = -4\frac{e_3 n}{2}(2-a)\cdot\frac{-1}{2n}[e_1 e_3(2-a)b + e_2 c] \tag{13}$$

$$= e_3(2-a)[e_1 e_3(2-a)b + e_2 c].$$

By Lemma B at least one of (12) and (13) is less than or equal 1, numerically. But θ is irrational; hence b in (3) is neither 0 nor 1. Therefore (a) or (b) of Lemma B must be less than 1, and the theorem is completely proved.

THEOREM 2. *There exist two relatively prime integers m, n such that*

$$|(\alpha m + \beta n)(\gamma m + \delta n)| < 1, \tag{14}$$

if $\alpha\delta - \beta\gamma = 1$. If α/β is irrational, infinitely many such pairs exist and $|\alpha m + \beta n|$ can be made arbitrarily small and at the same time $|\gamma m + \delta n|$ can be made arbitrarily large.

PROOF. We consider two cases.

CASE 1. α/β is rational. Then $\beta/\alpha = -(m/n)$, where m and n are relatively prime integers. Hence

$$\alpha m + \beta n = 0, \tag{15}$$

and the theorem is true.

CASE 2. α/β is irrational. We write

$$(\alpha x + \beta y) = \alpha\left(x + \frac{\beta}{\alpha}y\right) = \alpha(x - \theta y), \tag{16}$$

where θ is irrational. Let m/n be a convergent to θ. Then, from (3),

$$\frac{m}{n} - \theta = e_1 \frac{b}{n^2}. \tag{17}$$

Hence

$$m - \theta n = \frac{e_1 b}{n}. \tag{18}$$

Equation (18) shows that $|\alpha m + \beta n|$ can be made as small as we please for large enough n.

Since $\beta/\alpha = -\theta$, we obtain

$$\frac{m}{n} + \frac{\beta}{\alpha} = \frac{e_1 b}{n^2}, \tag{19}$$

which, solved for m, gives

$$m = -\frac{\beta}{\alpha} n + \frac{e_1 b}{n}. \tag{20}$$

Therefore

$$\begin{aligned} |\gamma m + \delta n| &= \left|\gamma\left(-\frac{\beta}{\alpha} n + \frac{e_1 b}{n}\right) + \delta n\right| \\ &= \left|\left(\delta - \frac{\beta}{\alpha}\gamma\right) n + \frac{e_1 b \gamma}{n}\right| = \left|\frac{n}{\alpha}\left(1 + \frac{e_1 b \alpha \gamma}{n^2}\right)\right|. \end{aligned} \tag{21}$$

The right-hand side of (21) can be made arbitrarily large for large enough n. If we choose m/n in (17) so that $(e_1 b\alpha\gamma)/n < 0$, (16), (18), and (21) give

$$|(\alpha m + \beta n)(\gamma m + \delta n)| < \frac{\alpha}{n} \cdot \frac{n}{\alpha} = 1. \tag{22}$$

Hence, since there are infinitely many convergents to θ, the theorem is completely proved.

THEOREM 3. *If $\xi = \alpha x + \beta y$ and $\eta = \gamma x + \delta y$ are two linear forms, and if $\alpha\delta - \beta\gamma = 1$, and if ξ_0 and η_0 are any two real numbers, integers x_1, y_1 can always be found such that*

$$|(\xi - \xi_0)(\eta - \eta_0)| \leqq \frac{1}{4}. \tag{23}$$

If α/β is irrational $|(\xi-\xi_0)|$ can be made arbitrarily small at the same time that (23) is satisfied.

PROOF. We write

$$\xi = \alpha x + \beta y, \qquad \eta = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma = 1. \tag{24}$$

By Theorem 2, two relatively prime integers m, n exist such that

$$\alpha m + \beta n = \lambda, \qquad \gamma m + \delta n = \mu, \qquad |\lambda\mu| < 1. \tag{25}$$

Without loss of generality we can assume that $\mu > 0$. Then we have

$$\lambda\mu = e_1 b, \tag{26}$$

where $e_1^2 = 1$, and $b < 1$. We define N and k by means of the equation

$$\mu\xi_0 - \lambda\eta_0 = N + k, \tag{27}$$

where N is an integer and $|k| \leqq 1/2$. Say that

$$k = \frac{e_2 c}{2}, \tag{28}$$

where $e_2^2 = 1$, and $c \leqq 1$. We can now write

$$\begin{aligned} \mu\xi - \lambda\eta = N &= (\gamma m + \delta n)(\alpha x + \beta y) - (\alpha m + \beta n)(\gamma x + \delta y) \\ &= nx - my. \end{aligned} \tag{29}$$

By Lemma A, $nx - my = N$ has a solution in integers x_0, y_0. Then all solutions of $nx - my = N$ are represented by

$$x_i = x_0 - im, \qquad y_i = y_0 - in, \tag{30}$$

where i assumes all integral values. Hence, since

$$\gamma x_i + \delta y_i = \gamma(x_0 - im) + \delta(y_0 - in) = (\gamma x_0 + \delta y_0) - i\mu, \tag{31}$$

it follows that i, an integer, can always be determined so that

$$\gamma x_i + \delta y_i - \eta_0 = \frac{e_3 a\mu}{2}, \tag{32}$$

where $e_3^2 = 1$, and $a \leqq 1$. Supposing i so determined and denoting $\alpha x_i + \beta y_i$ and $\gamma x_i + \delta y_i$ by ξ_1 and η_1, respectively, we get

$$\eta_1 - \eta_0 = \frac{e_3 a \mu}{2}. \tag{33}$$

Substituting ξ_1 for ξ and η_1 for η in (29), then subtracting (27) and using (28) and (33), we obtain

$$\mu(\xi_1 - \xi_0) = \lambda\eta_1 + N - \lambda\eta_0 - N - k = \lambda(\eta_1 - \eta_0) - \frac{e_2 c}{2}. \tag{34}$$

Therefore

$$(\xi_1 - \xi_0) = \frac{\lambda}{\mu} \cdot \frac{e_3 a}{2} \mu - \frac{e_2 c}{2\mu} = \frac{1}{2\mu}(e_1 e_3 a b - e_2 c). \tag{35}$$

Similarly, for $\eta_2 = \eta_1 - e_3\mu$, and $\xi_2 = \xi_1 - e_3\lambda$, we get

$$\eta_2 - \eta_0 = -\frac{e_3\mu}{2}(2 - a), \tag{36}$$

and

$$\xi_2 - \xi_0 = -\frac{1}{2\mu}[e_1 e_3(2 - a)b + e_2 c]. \tag{37}$$

For the product of (33) and (35) we have

$$4(\xi_1 - \xi_0)(\eta_1 - \eta_0) = e_3 a(e_1 e_3 a b - e_2 c), \tag{38}$$

while the result of multiplying (36) and (37) is

$$4(\xi_2 - \xi_0)(\eta_2 - \eta_0) = e_3(2 - a)[e_1 e_3(2 - a)b + e_2 c]. \tag{39}$$

If α/β is rational, we see from (26) and the proof of Case 1, Theorem 2, that $b = 0$ in (38); (38) is therefore less than or equal to 1, numerically, and therefore (23) is true. If α/β is irrational, by Lemma B at least one of (38) and (39) is less than or equal 1 numerically, while both (35) and (37) can be made arbitrarily small. Hence the theorem is completely proved.

STANFORD UNIVERSITY

SOME OBSERVATIONS ON THE THEORY OF FOURIER TRANSFORMS

BY E. HILLE, A. C. OFFORD, AND J. D. TAMARKIN†

1. *From a Letter by Offord.* On pages 770–771 of your paper,‡ *On the theory of Fourier transforms*, you prove the following result.

LEMMA. *If $g(t)$ is in L_p, $1<p<\infty$, and if*

$$g(s, a) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{\sin a(s-t)}{s-t}\,dt,$$

then

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} |\,g(s) - g(s, a)\,|^p ds = 0.$$

You make no use of your hypothesis $p\leqq 2$ in this part of your paper. Now consider Berry's paper, *The Fourier transform identity theorem.*§

Write

$$G(s, a) = (2\pi)^{-1/2}\int_{-a}^{a} g(t)e^{-ist}dt,$$

and suppose $g(t)$ has a Fourier transform $G(s)$ in L_q, $1<q<\infty$, that is,

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} |\,G(s, a) - G(s)\,|^q ds = 0.$$

THEOREM A (Berry). *If $g(s)$ has a Fourier transform $G(s)$ in L_q, $1<q<\infty$, and if $G(s)$ has a Fourier transform $\mathfrak{G}(s)$ in L_p, $1<p<\infty$, then $\mathfrak{G}(s)=g(-s)$.*

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† The present note contains excerpts from a letter by Offord to Tamarkin, and from a reply to this letter by Hille and Tamarkin. Before knowing the contents of Offord's letter Hille arrived independently at some of Offord's conclusions, as well as at extensions in other directions.

‡ E. Hille and J. D. Tamarkin, this Bulletin, vol. 39 (1933), pp. 768–774.

§ Annals of Mathematics, (2), vol. 32 (1931), pp. 227–232.

This is a uniqueness theorem. The methods of your paper seem to give the existence theorem.

THEOREM B. *If $g(s)$ belongs to L_p, $1<p<\infty$, and if it has a Fourier transform $G(s)$ in some L_q, $1<q<\infty$, then $g(-s)$ is the Fourier transform in L_p of $G(s)$.*

We have

$$(*) \qquad G(s) = (2\pi)^{-1/2} \operatorname*{l.i.m.}_{a\to\infty} \int_{-a}^{a} g(t)e^{-ist}dt \qquad (\text{in } L_q).$$

Hence

$$\begin{aligned}(2\pi)^{-1/2}\int_{-b}^{b} G(u)e^{isu}du &= (2\pi)^{-1}\lim_{a\to\infty}\int_{-b}^{b} e^{isu}du\int_{-a}^{a} g(t)e^{-iut}dt \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} g(t)dt\int_{-b}^{b} e^{iu(s-t)}du \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{\sin b(s-t)}{s-t}\,dt.\end{aligned}$$

The desired result now follows from your lemma.

It is possible to go a little further than this.

THEOREM C. *If $g(s)$ belongs to L_p, $1<p<\infty$, and if*

$$\int_{-\infty}^{\infty} |G(s, a)|^q ds \leqq M, \qquad (1 < q < \infty),$$

for all a, then $g(s)$ has a Fourier transform $G(s)$ in L_q and $g(-s)$ is the Fourier transform in L_p of $G(s)$.

This follows from Theorems 1 and 7 of my paper, *On Fourier transforms*, III.† Theorem C is a corollary of the following theorem.

THEOREM D.‡ *Let $f(u)$ be integrable over every finite range and let*

$$F(x, a) = (2\pi)^{-1/2}\int_{-a}^{a} f(u)e^{-ixu}du.$$

† To appear in the Transactions of this Society.

‡ For the corresponding theorem when $p=\infty$ see A. C. Offord, *On Fourier transforms*, Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 197–216.

Let

$$(**) \qquad \int_{-\infty}^{\infty} |F(x, a)|^p dx \leqq M^p,$$

where $1 < p < \infty$. *Then* $F(x, a)$ *converges* $(C, 1)$ *in* L_p *to a function* $F(x)$ *and*

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u) e^{ixu} du$$

almost everywhere.

It is not assumed that $f(x)$ belongs to a Lebesgue class over $(-\infty, \infty)$. Let us write

$$F_1(x, a) = (2\pi)^{-1/2} \int_{-a}^{a} \left(1 - \frac{|u|}{a}\right) f(u) e^{-ixu} du.$$

If we can show that (**) implies

$$(***) \qquad \int_{-\infty}^{\infty} |F_1(x, a)|^p dx \leqq M^p,$$

then Theorem D will follow from Theorems 1 and 7 of my paper. Now

$$F_1(x, A) = A^{-1} \int_0^A F(x, a) da.$$

The desired result (***) follows from this by an easy application of Hölder's inequality.

2. *From an Answer by Hille and Tamarkin.*† The fact that our lemma above holds for any $p > 1$, and not only for $1 < p \leqq 2$, was observed by ourselves,‡ and also was called to our attention by

† It should be observed that results essentially equivalent to our Theorems 1, 3, 5 of the present note are contained in Offord's papers referred to above, although Offord does not use the notions of Fourier transforms in the sense of our Definitions 1 and 2. The paper of Offord mentioned in the preceding footnote was not available to us at the time when the present reply was being written.

‡ *A remark on Fourier transforms and functions analytic in a half plane*, Compositio Mathematica, vol. 1 (1934), pp. 98–102, especially p. 100. It should be stated that a result analogous to our lemma was obtained at a much earlier date by A. Berry, in an unpublished paper of his.

A. Zygmund. It is obvious that the proof of Theorem B does not utilize the full force of the condition (*). The only property of $g(s)$ which comes into play is that

$$\int_{-b}^{b} G(u)e^{isu}du = \lim_{a\to\infty} (2\pi)^{-1/2} \int_{-b}^{b} e^{isu}du \int_{-a}^{a} g(t)e^{-iut}dt$$

$$= \lim_{a\to\infty} \int_{-b}^{b} g(u, a)e^{isu}du$$

for all $b>0$ and all real s. This observation leads naturally to the following definition.

DEFINITION 1. *Let $f(x)$ be integrable over every finite range. Let*

$$f(x, a) = (2\pi)^{-1/2} \int_{-a}^{a} f(t)e^{-itx}dt. \tag{1}$$

If there exists a sequence $\{a_n\}$, $a_n \uparrow \infty$ such that, for all $b>0$ and all real x,

$$\lim_{n\to\infty} \int_{-b}^{b} f(u, a_n)e^{ixu}du = \int_{-b}^{b} F(u)e^{ixu}du, \tag{2}$$

where $F(x)$ is integrable over every finite range, then $F(x)$ is called a Fourier transform of $f(x)$.

This definition finds its justification in the following theorem.

THEOREM 1. *If $f(x) \subset L_p$, $1 \leqq p < \infty$, and if $f(x)$ has a Fourier transform $F(x)$, then $F(x)$ is uniquely determined by $f(x)$, that is, does not depend on the choice of the particular sequence $\{a_n\}$ for which (2) is satisfied. If $p>1$, then $f(-x)$ is the Fourier transform in L_p of $F(x)$.*

Indeed from (2) and (1) we have

$$(2\pi)^{-1/2} \int_{-b}^{b} F(u)e^{ixu}du = (2\pi)^{-1} \lim_{n\to\infty} \int_{-b}^{b} e^{ixu}du \int_{-a_n}^{a_n} f(t)e^{-itu}dt$$

$$= (2\pi)^{-1} \lim_{n\to\infty} \int_{-a_n}^{a_n} f(t)dt \int_{-b}^{b} e^{iu(x-t)}du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin b(x-t)}{x-t} dt.$$

The right-hand member here does not depend on $\{a_n\}$. Hence, if $F_1(x)$ is any other Fourier transform of $f(x)$ in the sense of Definition 1, then again for all $b>0$ and all x,

$$(2\pi)^{-1/2}\int_{-b}^{b} F_1(u)e^{ixu}du = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{\sin b(x-t)}{x-t}\,dt,$$

which implies $F(x)=F_1(x)$, except perhaps for a set of values of x of measure zero. Consequently $F(x)$ is determined uniquely by $f(x)$, if it exists at all. The fact that $f(-x)$ is the Fourier transform in L_p of $F(x)$ if $p>1$ follows immediately from our Lemma.

Condition (2) is obviously satisfied if the set of functions $f(x, a)$, $a\geqq 0$, belongs to some L_q, $1<q<\infty$, and if the sequence $\{a_n\}$ can be selected in such a way that the sequence of functions $f(x, a_n)$ converges weakly in L_q to a weak limit $F(x)$. This will certainly be the case when the set $f(x, a)$, $a\geqq 0$, is bounded in L_q, that is, when

$$\int_{-\infty}^{\infty} |f(x, a)|^q \leqq M^q.$$

If this condition is satisfied, it is readily seen that there exists a function $F(x)\subset L_q$ such that $f(x, a)$ converges weakly to $F(x)$ as $a\to\infty$. This function $F(x)$ therefore appears as the Fourier transform of $f(x)$, while $f(-x)$ is the Fourier transform in L_p of $F(x)$. Your Theorem D furnishes a sharper result, however, namely, that $f(x, a)$ converges (strongly) to $F(x)$ in L_q and this even without the assumption that $f(x)\subset L_p$. In other words, the boundedness in L_q of the set $f(x, a)$ implies its convergence in L_q to the Fourier transform $F(x)$ of $f(x)$, and vice versa. The last statement is obvious.

Upon setting $\Phi(x)=\int_0^x F(t)dt$ we may rewrite (2) in the form

$$\lim_{n\to\infty}\int_{-b}^{b} f(u, a_n)e^{ixu}du = \int_{-b}^{b} e^{ixu}d\Phi(u).$$

This immediately leads to a further generalization of the notion of the Fourier transform.

DEFINITION 2. *Let $f(x)$ be integrable over every finite range. Let $\Phi(x)$ be defined and finite for all values of x and integrable over*

every finite range. Let $\int_\alpha^\beta e^{ixu}d\Phi(u)$ *be the generalized Stieltjes integral defined by*

$$(3)\qquad \int_\alpha^\beta e^{ixu}d\Phi(u) = e^{ix\beta}\Phi(\beta) - e^{ix\alpha}\Phi(\alpha) - ix\int_\alpha^\beta e^{ixu}\Phi(u)du.$$

If there exists a sequence $\{a_n\}$, $a_n \uparrow \infty$, *such that for all* $b>0$ *and for all real* x,

$$(4)\qquad \lim_{n\to\infty}\int_{-b}^{b} f(u, a_n)e^{ixu}du = \int_{-b}^{b} e^{ixu}d\Phi(u),$$

then $\Phi(x)$ *is called a Fourier transform of order* 1 *of* $f(x)$.†

Definition 2 looks very much the same as definitions used by Hahn, Wiener, and Bochner.‡ There is one essential difference, however, due to the fact that Definition 2 does not a priori specialize the behavior of $f(x)$ at infinity, while other definitions do so. From the example at the end of this note it will be seen that there exist cases in which Definition 2 can be applied while no Bochner transform of any order can be defined.

Definition 2 is justified not only by the fact that it generalizes Definition 1 to which it reduces when $\Phi(x)$ is absolutely continuous, but also by the following theorem.

THEOREM 2. *If* $f(x)$ *is integrable over every finite range and if*

$$\lim_{a\to\infty}\frac{1}{\pi}\int_{-a}^{a} f(t)\,\frac{\sin b(x-t)}{x-t}\,dt$$

exists for each x, *then the Fourier transform of order* 1 *of* $f(x)$ *is uniquely determined up to an additive constant whenever it exists at all.*

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† This definition closely resembles the one we used in our note, *On a theorem of Paley and Wiener*, Annals of Mathematics, (2), vol. 34 (1933), pp. 606–614, especially pp. 609–611. We intend to discuss the relationship between these two definitions elsewhere. A definition identical to that of our Annals paper was successfully used in a recent paper by Verblunsky, *Trigonometric integrals and harmonic functions*, Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 1–48.

‡ See Wiener, *Generalized harmonic analysis*, Acta Mathematica, vol. 55 (1930), pp. 117–258, especially p. 159; Bochner, *Vorlesungen über Fouriersche Integrale*, 1932, Chapter 6: other references are also given there.

In particular, if $f(x) \subset L_p$, $1 \leqq p < \infty$, then the Fourier transform of order 1 of $f(x)$ is given by

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} f(t)dt, \qquad (x \neq 0). \dagger \tag{5}$$

Furthermore, if $p > 1$, the function

$$F(x, b) = (2\pi)^{-1/2} \int_{-b}^{b} e^{ixu} d\Phi(u) \tag{6}$$

converges in L_p to $f(x)$ as $b \to \infty$.

Applying the same argument as in the proof of Theorem 1, we now have

$$(2\pi)^{-1/2} \int_{-b}^{b} e^{ixu} d\Phi(u) = \lim_{a \to \infty} \frac{1}{\pi} \int_{-a}^{a} f(t) \frac{\sin b(x - t)}{x - t} dt. \tag{7}$$

Hence if there exists another Fourier transform of order 1 of $f(x)$, say $\Phi_1(x)$, we must have, for all $b > 0$ and all x,

$$\begin{aligned} 0 &= \int_{-b}^{b} e^{ixu} d\Psi(u) \\ &= \Psi(b)e^{ixb} - \Psi(-b)e^{-ixb} - ix \int_{-b}^{b} e^{ixu} \Psi(u) du, \end{aligned}$$

where $\Psi = \Phi - \Phi_1$. For $x = 0$ this gives $\Psi(b) = \Psi(-b)$. Upon setting $\Psi_0(u) = \Psi(u) - \Psi(b)$, we now have

$$\int_{-b}^{b} e^{ixu} \Psi_0(u) du = 0.$$

Hence $\Psi_0(u) = 0$, $\Psi(u) = \Psi(b)$ almost everywhere in $(0, b)$. Since b is arbitrary we conclude that $\Psi(x) = \text{const.}$ for all $x \neq 0$.

Now assume that $f(x) \subset L_p$, $1 \leqq p < \infty$. Then in (7) we may replace $\lim_{a \to \infty} \int_{-a}^{a}$ by $\int_{-\infty}^{\infty}$, the integral being absolutely convergent.

The fact that $F(x, b) \to f(x)$ in L_p as $b \to \infty$, if $p > 1$, follows then from our lemma. Hence it remains only to prove that the

† The value of $\Phi(x)$ at $x = 0$ is irrelevant since it can be modified arbitrarily without changing the value of $F(x, b)$.

Fourier transform of order 1 of $f(x)$ exists and is given by (5). To do so we evaluate $F(x, b)$ with

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} f(t)dt,$$

which is a continuous function of x. Thus we get

$$\begin{aligned} F(x, b) &= (2\pi)^{-1/2} \int_{-b}^{b} e^{ixu} d\Phi(u) \\ &= (2\pi)^{-1/2} \left[e^{ixu}\Phi(u)\right]_{u=-b}^{u=b} - (2\pi)^{-1/2} ix \int_{-b}^{b} e^{ixu}\Phi(u)du \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} \left[(e^{ib(x-t)} - e^{-ib(x-t)} - e^{ibx} + \frac{e^{-ibx}}{-it} f(t)\right]dt \\ &\quad - \frac{ix}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)}{-it} dt \int_{-b}^{b} (e^{iu(x-t)} - e^{ixu})du \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin b(x-t)}{x-t} dt = \lim_{a\to\infty} (2\pi)^{-1/2} \int_{-b}^{b} f(u, a)e^{ixu}du, \dagger \end{aligned}$$

which is precisely relation (4) of Definition 2.

Theorem 2 shows that a function $f(x) \subset L_p$, $1 \leqq p < \infty$, always has a Fourier transform of order 1, which is a continuous function of x. A natural question arises whether every $f(x) \subset L_p$ has a Fourier transform (in the sense of Definition 1). We shall prove that in general this is not the case, if $p > 2$.

THEOREM 3. *A necessary and sufficient condition that a function $f(x)$ should have a Fourier transform in the sense of Definition* 1, *is that the continuous function*

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-ixt} - 1}{-it} f(t)dt \tag{8}$$

be absolutely continuous.

This result is easily derived by combining Definition 1 and Theorems 1 and 2. The question of the existence of a Fourier transform of $f(x)$ is thus reduced to the discussion of the abso-

† See an analogous computation in Wiener, loc. cit.

lute continuity of $\Phi(x)$ as given by (8). It is well known that if $1<p\leqq 2$, every $f(x)\subset L_p$ has a Fourier transform in $L'_{p'}$, $p'=p/(p-1)$. This Fourier transform must coincide with the Fourier transform of Definition 1. Hence $\Phi(x)$ is absolutely continuous when $1<p\leqq 2$. This result can also be verified directly.

Now assume $p>2$. Consider L_p as a metric vector space with the usual definition of its metric. Referring to a previous result of ours† we see that the set of functions of L_p for which $\Phi(x)$ is of bounded variation in any interval $(0, A)$, $A>0$, is of the first category in L_p. By modifying slightly the argument there used we might even replace the interval $(0, A)$ by an arbitrary interval, no matter how small. Consequently we may state the following result.

THEOREM 4. *If $2<p<\infty$, the set of functions of L_p which possess a Fourier transform in the sense of Definition* 1 *is of the first category in L_p. If $1\leqq p\leqq 2$ every $f(x)\subset L_p$ has a Fourier transform, indeed a Fourier transform in $L_{p'}$.*

It is of interest to exhibit a sufficient criterion for the existence of the Fourier transform of $f(x)\subset L_p$, $p>2$, which is more general than that of your Theorem C which requires the boundedness in L_q of the set $f(x, a)$, $a\geqq 0$.

THEOREM 5. *Let $f(x)\subset L_p$, $1\leqq p<\infty$, and let, for all $a\geqq a_0\geqq 0$ and for almost all x,*

$$|f(x, a)| \leqq P(x), \tag{9}$$

where $P(x)$ is a positive measurable function, integrable over every finite range. Then $f(x)$ has a unique Fourier transform in the sense of Definition 1.

In view of Theorem 3 it suffices to establish the absolute continuity of $\Phi(x)$. Now

$$\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{e^{-itx}-1}{-it}f(t)dt = \lim_{a\to\infty}\int_0^x f(t, a)dt,$$

and (9) shows that the set of functions $\int_0^x(t, a)dt$ is absolutely

† E. Hille and J. D. Tamarkin, *On moment functions*, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 902–908, especially p. 905.

continuous over every fixed finite range of x, uniformly in a. Hence the limit function $\Phi(x)$ is also absolutely continuous.

A striking example of an application of Theorem 5 is given by the following function†

$$f(x) = \begin{cases} x^{-\alpha}e^{ix\log x}, & x \geqq 2, \\ 0 \quad , & x < 2, \end{cases} \qquad (0 < \alpha \leqq 1/2). \tag{10}$$

Here $f(x, a)$ converges to a continuous function $F(x)$, uniformly over every finite interval. However, for large positive x,

$$F(x) = \exp\{i(\pi/4 - e^{x-1}) + (1/2 - \alpha)(x - 1)\}(1 + o(1)); \tag{11}$$

hence $F(x)$ does not belong to any Lebesgue class over $(-\infty, \infty)$ Nevertheless the condition of Theorem 5 is satisfied and $F(x)$ is the Fourier transform of $f(x)$ in the sense of Definition 1, while $f(-x)$ is the Fourier transform of $F(x)$ in L_p for $p > 1/\alpha$. This is not the worst possible behavior that $F(x)$ may exhibit under such circumstances. We have merely to replace the term $\log x$ in (10) by repeated logarithms of sufficiently high order to introduce iterated exponentials of arbitrarily high order in (11). In all cases $F(x)$ has a Fourier transform in L_p, hence also in the sense of Definitions 1 and 2. However, the integral $\int_{-\infty}^{\infty} |F(x)|/(1+|x|^k)dx$ does not exist, no matter what $k > 0$ is. Hence the Wiener-Bochner theory can not be applied to $F(x)$.

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† E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, III, Mathematische Annalen, vol. 108 (1933), pp. 525–577, especially p. 575. The factor $(2\pi)^{1/2}$ should be crossed out in formula (18.22).

A THEOREM CONCERNING LOCALLY PERIPHERALLY SEPARABLE SPACES*

BY F. B. JONES

Alexandroff has shown that a connected metric space is completely separable provided it is locally completely separable.† It is the purpose of this paper to establish a similar theorem for connected, locally connected metric spaces.

DEFINITION. *A space is locally peripherally separable provided that, if P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is separable.*

THEOREM. *A connected, locally connected, locally peripherally separable metric space is completely separable.*

PROOF. Suppose that n is a fixed positive integer. Let G denote the collection of all domains (to be called regions) of diameter $1/n$ or less which are peripherally separable. Since space is locally peripherally separable, it is evident that G covers space. It will now be shown that G contains a countable subcollection covering space. For each point X of space let n_x denote the largest integer such that no region of G contains a circular domain with center at X and radius greater than or equal to $1/n_x$. Let D_1 denote some region of G. For each integer i let M_{1i} denote the set of all points X of the boundary β_1 of D_1 such that $n_x=i$. Since space is metric and β_1 is separable , β_1 is completely separable, and there exists in M_{1i} a countable point set N_{1i} which is everywhere dense in M_{1i}. Now for each point X of N_{1i} let R_x denote a region of G containing a circular domain with center at X and radius $1/(i+1)$. The sum of these regions R_x forms a domain Q_{1i} covering M_{1i}, and $\sum_{i=1}^{\infty}Q_{1i}$ is a domain covering β_1.

Let $D_2=D_1+\sum_{i=1}^{\infty}Q_{1i}$. Then D_2 contains $D_1+\beta_1$. Since space is locally connected, every point of the boundary β_2 of D_2 either belongs to the boundary of some region R_x or is a limit point of

* Presented to the Society, June 20, 1934.

† Paul Alexandroff, *Über die Metrization der im kleinen kompakten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 294–301.

the sum of the boundaries of these regions R_x.* Furthermore, there are only countably many regions R_x. Hence, β_2 is separable. It is also to be noted that D_2 is the sum of countably many regions of G.

This process may be continued. Thus there exist a sequence $D_1, D_2, D_3, \cdots$ of domains and for each integer m a sequence $N_{m1}, N_{m2}, N_{m3}, \cdots$ of point sets, so that if, for each integer m, β_m denotes the boundary of D_m, and, for each pair of integers m and i, M_{mi} denotes the set of all points X of β_m such that $n_x = i$, then

(0) D_m is the sum of countably many regions of G,

(1) D_m contains $D_{m-1}+\beta_{m-1}$,

(2) $\beta_m = \sum_{i=1}^{\infty} M_{mi}$,

(3) N_{mi} is a countable subset of M_{mi} which is everywhere dense in M_{mi},

(4) if X is a point of N_{mi}, D_{m+1} contains a circular domain with center at X and radius $1/(i+1)$, and

(5) if R is a region of G containing a point X of N_{mi}, no circular domain V_x lying in R with center at X has a radius greater than or equal to $1/i$.

Suppose now that there is some point of space not belonging to any D_m. Then, since space is connected, there exists a point O on the boundary of $\sum D_m$. Hence, it follows from (1), (2), and (3) that O is a sequential limit point of a sequence of points $X_1, X_2, X_3, \cdots$, where for each m, X_m belongs to N_{mi} for some i. Now, let R denote a region of G containing a circular domain V_0 with center at O and radius r. For each point X_m of the sequence $X_1, X_2, X_3, \cdots$ let V_{X_m} denote a circular domain with center at X_m and radius $r/2$. There exists a number k such that if $m>k$, then V_{X_m} lies in R. However, since O is not covered by $\sum D_m$ and N_{mi} contains X_m, it follows from (4) that i increases with m. Hence, for some point X_m, $(m>k)$, $r/2$ is greater than $1/i$. This contradicts (5); so $\sum D_m$ covers space. By (0), D_m is the sum of countably many regions of G, and therefore $\sum D_m$ is the sum of the regions of a countable subcollection G' of G. Hence G' covers space.

Now, for each integer n, let G_n denote the collection of periph-

* R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13, 1932, Theorem 2(b), Chapter II, p. 91.

erally separable domains of diameter $1/n$ or less. By the above argument each G_n contains a countable subcollection G_n' covering space. Hence, space is completely separable and the theorem is established.

THE UNIVERSITY OF TEXAS

A REDUCED SET OF POSTULATES FOR ABSTRACT HILBERT SPACE*

BY A. E. TAYLOR

1. *Introduction.* An abstract Hilbert space is a normed linear space, or *vector space*, of infinite dimensionality, with a norm based on a Hermitian *inner product*, defined for all pairs of elements in the space. The space is, moreover, separable and complete according to this norm. The usual postulate system for Hilbert space, which was first stated abstractly by J. von Neumann, consists of five groups of postulates, or nineteen in all.†

The purpose of the present paper is to demonstrate the redundancy‡ of a number of the postulates, and to present a system of eleven independent postulates for a normed linear space with a Hermitian inner product. The adjunction of three more postulates, each of which is independent of the first eleven and the remaining two, then gives us a system which is equivalent to that of von Neumann, that is, it defines an abstract Hilbert space, and it is categorical.

A special feature of this postulate system is that the abstract relation called equality, and denoted, as usual, by the symbol $=$, enters on an equal footing with the operations defined in the space.§ Three of the eleven postulates are concerned with this

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* Presented to the Society, December 1, 1934.

† J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, 1932. pp. 19–24; M. H. Stone, *Linear Transformations in Hilbert Space*, Colloquium Publications of this Society, vol. 15, pp. 2–4.

‡ Some of these redundancies were noted simultaneously by a fellow-student, Mr. Ivar Highberg, and myself.

§ The postulational treatment of equality in vector spaces was suggested by A. D. Michal in a critique of postulate systems. See this Bulletin, vol. 39 (1933), Abstract No. 339.

relation, giving its properties in connection with the operations. It is then demonstrated that the equality relation has the usual properties of an equivalence relation: reflexiveness, symmetry, and transitivity.

2. *A Normed, Linear Space.* Let K be a class of elements $x, y, \cdots$; and let C be the class of complex numbers $a, b, c, \cdots$. Let there be given in K a binary relation, called equality, and denoted by $=$, such that given an ordered pair of elements x, y from K, then either x bears the relation to y $(x=y)$, or it does not $(x \neq y)$. Let $+$ be the symbol of a binary operation, or rule of combination, defined throughout K. That is, to each ordered pair of elements x, y from K, there corresponds a unique entity, called their sum, and denoted by $x+y$. Let $\cdot$ be the symbol of a binary operation defined on the composite of C and K; that is, given $a \epsilon C$ and $x \epsilon K$, there is uniquely determined an entity, denoted by $a \cdot x$, or simply ax. Let $(\ ,\ )$ be the symbol of a binary operation, called the inner product, defined for all ordered pairs $x, y \epsilon K$, and yielding a unique entity, written (x, y).

The universe of discourse composed of the class K, the relation $=$, and the three operations $+, \cdot, (\ ,\ )$, is to be governed by the following eleven postulates, and it then forms a special kind of normed, linear space, as will be proved in §3.

1.0 *The class K contains at least one element.*

1.1 *If $x, y \epsilon K$, then $x+y \epsilon K$.*

1.2 *If $a \epsilon C$ and $x \epsilon K$, then $a \cdot x \epsilon K$.*

1.3 *If $x, y, (-1) \cdot y, x+(-1) \cdot y \epsilon K$, and if for each element $u \epsilon K$, $[x+(-1)y]+u=u$, then $x=y$.*

1.4 *If $x, y \epsilon K$, then $(x, y) \epsilon C$.*

1.5 *If $x, y, z, x+y \epsilon K$, and $(x+y, z), (x, z), (y, z) \epsilon C$, then $(x+y, z)=(x, z)+(y, z)$.*

1.6 *If $x, y \epsilon K$, and $(x, y), (y, x) \epsilon C$, then $(x, y)=\overline{(y, x)}$.*

1.7 *If $x, y, ax \epsilon K$, and $(ax, y), (x, y) \epsilon C$, then $(ax, y)=a(x, y)$.*

1.8 *If $x \epsilon K$ and $(x, x) \epsilon R$, then $(x, x) \geqq 0$, (R denotes the class of real numbers).*

1.9 *If $x \epsilon K$ and $(x, x)=0$, then $x+y=y$ for each y in K.*

1.10 *If $x, y \epsilon K$, and $x \neq y$, and if u is an arbitrary element in K such that $(x, u), (y, u) \epsilon C$, then $(x, u) \neq (y, u)$.*

This statement of the postulates is such as to make each one intelligible independently of the others. Thus, if 1.1, 1.2, and

1.4 hold, the remaining postulates may be greatly abbreviated. We also remark a double use of the symbols $=, +, \cdot$. No confusion will arise; the context suffices to make the meaning clear.

3. *General Theorems.* This section will be devoted to proving that the universe of discourse defined in §2 is a normed, linear space.* Questions of consistency and independence are discussed in §4.

DEFINITION. If $x \in K$, then $(-1)x \in K$, by 1.2, and we call this latter element $-x$ $(-x \equiv (-1)x)$. Similarly we define $x-y \equiv x+(-1)y$.

The following theorems are stated without proof.†

THEOREM 1. *If* $x, y \in K$, *and* $a \in C$, *then* $(x, ay) = \bar{a}(x, y)$.

THEOREM 2. *If* $x, y, z \in K$, *then* $(x, y+z) = (x, y)+(x, z)$.

THEOREM 3. *If* $x, y, u, v \in K$, *and* $a, b, c, d \in C$, *then*
$(ax + by, cu + dv) = a\bar{c}(x, u) + b\bar{c}(y, u) + a\bar{d}(x, v) + b\bar{d}(y, v)$.

DEFINITION. An element $z \in K$ such that $z+y=y$ for each $y \in K$, is called a zero element.

THEOREM 4. *There exists at least one zero element in* K. *If* x *is any element in* K, *then* $0 \cdot x$ *is a zero element, and* $x-x$ *is also.*

PROOF. By 1.0, there exists at least one element $x \in K$. Therefore $0 \cdot x \in K$, by 1.2. By 1.9, any element y such that $(y, y) = 0$ is a zero element. But if x is arbitrary, $(0 \cdot x, 0 \cdot x) = 0$, by 1.7. Also $(x-x, x-x) = 0$, by Theorem 3. We shall presently prove that all zero elements are equal.

THEOREM 5. *If* $x, y \in K$, *then* $x+y=y+x$ (*Commutativity*). *If* $x, y, z \in K$, *then* $x+(y+z) = (x+y)+z$ (*Associativity*).

We shall prove the commutativity; the proof of associativity is quite similar, and will be left to the reader. We need merely show that $(x+y)-(y+x)$ is a zero element; the result then follows by postulate 1.3. Consider the inner product

$$((x + y) - (y + x), (x + y) - (y + x)).$$

* S. Banach, *Théorie des Operations Linéaires*, 1932, pp. 26–37 and p. 53. Banach calls them "espaces vectoriels normés."

† M. H. Stone, loc. cit., p. 4.

It is easily seen, on expanding this with the aid of Theorem 3, that the inner product is zero, and that therefore, by 1.9, $(x+y)-(y+x)$ is a zero element.

The next group of theorems establishes the familiar properties of the equality relation.

THEOREM 6. *If* $x, y \epsilon K$, *and* $x=y$, *then* $(x, x)=(x, y)=(y, x)=(y, y)$.

The proof is accomplished by replacing the arbitrary element u of postulate 1.10, first by x, and then by y. We thus obtain $(x, x)=(y, x)$, and $(x, y)=(y, y)$. But since (x, x) is real, it follows by 1.6 that $(y, x)=(x, y)$. Thus the theorem is proved.

THEOREM 7. *If* $x, y, u, v \epsilon K$, *and if* $x=y$ *and* $u=v$, *then* $x+u=y+v$.

PROOF. Consider the inner product

$$((x+u)-(y+v), (x+u)-(y+v)).$$

We find, upon expansion of this according to Theorem 3, that postulate 1.10 and Theorem 6 reduce the resulting expression to zero. Consequently the element $(x+u)-(y+v)$ is a zero element, and $x+u=y+v$. In quite similar fashion the following theorem is proved.

THEOREM 8. *If* $a, b \epsilon C$, *and* $x, y \epsilon K$, *and if* $a=b$ *and* $x=y$, *then* $ax=by$.

THEOREM 9. *The relation* $=$ *is an equivalence relation. That is, if* $x \epsilon K$, *then* $x=x$ (*reflexiveness*). *If* $x, y \epsilon K$, *and* $x=y$, *then* $y=x$ (*symmetry*). *If* $x, y, z \epsilon K$, *and* $x=y$ *and* $y=z$, *then* $x=z$ (*transitivity*).

PROOF. We have already seen that $x-x$ is a zero element (Theorem 4). It follows at once that $x=x$. To prove the symmetry we consider the element $y-x$ and the inner product $(y-x, y-x)$. On expanding, we find the relation:

$$(y-x, y-x)=(y, y)-(y, x)-(x, y)+(x, x).$$

The expression on the right vanishes, by Theorem 6. Hence, by postulates 1.9 and 1.3, $y=x$. Finally, let $x=y$ and $y=z$. Then

$$(x-z, x-z)=(x, x)-(z, x)-(x, z)+(z, z).$$

From Theorem 6 and postulate 1.6 we have the relations:

$$(x, x) = (y, y) = (z, z), \qquad (z, z) = (z, y) = (y, z),$$
$$(z, x) = (z, y) = (y, z), \qquad (x, z) = (z, x) = (z, y).$$

From these it follows that $(x-z, x-z)=0$, so that $x=z$.

Postulates 1.0, 1.1 and Theorems 4, 5, 7, 9 suffice to show that the elements of K form an abelian group with respect to addition, using the equality relation.* The consequences of this are summed up in the following theorem.

THEOREM 10. *The class K forms an abelian group with respect to addition.*

The equation $x+y=z$ is uniquely (to within equal elements) solvable for x, and indeed $x=z-y$. In particular, the zero element in K is unique (that is, all zero elements are equal); and to each $x \epsilon K$ corresponds a unique inverse, $-x$.

We shall denote the zero element in K by 0. This will occasion no ambiguity. The next group of theorems deals with the properties of multiplication by complex numbers.

THEOREM 11. *If $x \epsilon K$, then $1 \cdot x = x$. If $a \epsilon C$, and 0 is the zero element in K, then $a \cdot 0 = 0$.*

PROOF. Consider the element $1 \cdot x - x$; on expanding the inner product we find that $(1 \cdot x - x, 1 \cdot x - x) = 0$. Therefore $1 \cdot x = x$. Since 0 is the unique zero element, it may be written in the form $0 \cdot x$, where x is arbitrary. It follows that $(0, 0) = 0$. But $(a \cdot 0, a \cdot 0) = |a|^2 (0, 0) = 0$. Hence $a \cdot 0 = 0$.

THEOREM 12. *If $x, y \epsilon K$ and $a, b \epsilon C$, then $a(x+y) = ax+ay$, $(a+b)x = ax+bx$, $a(bx) = (ab)x$.*

PROOF. Each of the equalities is established by the method used in the proof of Theorem 5. For instance, by considering the element $a(bx) - (ab)x$, we show that its inner product with itself is zero, and hence that $a(bx) - (ab)x = 0$.

This completes the demonstration that K is a linear space. In fact, the system of postulates 1.0–1.10 is equivalent to the first two groups of postulates stated by von Neumann.†

* B. L. Van der Waerden, *Moderne Algebra*, 1930, vol. 1, p. 15.

† J. von Neumann, loc. cit., p. 19–21; von Neumann does not explicitly state any postulates for equality, but he uses all the properties which we have assumed or proved.

DEFINITION. By the *norm* of an element x we mean the non-negative number $(x, x)^{1/2}$; for convenience we shall write $\|x\| \equiv (x, x)^{1/2}$.

It is readily demonstrated that this norm has the usual properties. The proof is based on the Schwarz inequality; we summarize the results in the following theorem.*

THEOREM 13. *If $x, y \in K$, and $a \in C$, then*

$$|(x, y)| \leqq \|x\| \cdot \|y\|, \quad \|x\| \geqq 0, \quad \|0\| = 0,$$
$$\|x + y\| \leqq \|x\| + \|y\|, \quad \|ax\| = |a| \cdot \|x\|.$$

4. *Consistency and Independence of the Postulates.* Postulates 1.0–1.10 are satisfied in any abstract Hilbert space, and are therefore consistent. As a specific example consider the class H_0 of infinite one-rowed matrices of complex numbers $x = (x_1, x_2, \cdots)$ such that the series $\sum_{i=1}^{\infty} |x_i|^2$ is convergent. If $y = (y_1, y_2, \cdots)$ is an element of H_0, we define $x = y$ if and only if $x_i = y_i$, $(i = 1, 2, 3, \cdots)$; and define

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots),$$
$$a \cdot x = (ax_1, ax_2, \cdots), \quad (x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

This system forms a Hilbert space.† For the sake of uniformity, most of the independence examples have been formed from variations of this example. We present these examples below, with a brief explanation of each one; the details of verifying the postulates in each case are rather simple, and are, for the most part, left to the reader. The number of the example denotes the postulate which is *not* satisfied, thereby being proved independent.

EXAMPLE 1.0. Let K be any empty class. Then postulates 1.1–1.10 are satisfied vacuously.

EXAMPLE 1.1. Let M be a positive real number, and consider the class of all infinite one-rowed matrices of complex numbers $x = (x_1, x_2, \cdots)$ in which at most M elements of the matrix are distinct from zero. This is a sub-class of the class H_0, and the

* J. von Neumann, loc. cit., pp. 21–22; M. H. Stone, loc. cit. pp. 4–5.

† M. H. Stone, loc. cit., pp. 14–15.

relation and operations are defined as in H_0. All the postulates are satisfied save that of closure under addition.

EXAMPLE 1.2. Consider the sub-class of H_0 in which all the elements of the matrices are *real* numbers, and let the relation and operations be defined as in H_0. All the postulates are satisfied save that of closure under multiplication by complex numbers.

EXAMPLE 1.3. Consider the class of complex-valued functions of the form

$$F + \sin nx, \qquad (-\infty < x < \infty),$$

where F is a complex number, and n is a non-negative integer. Then, if $f = F + \sin nx$, and $g = G + \sin mx$, we define

$$\begin{aligned} f &= g, \text{ if and only if } F = G \text{ and } n = m, \\ f + g &= F + G + \sin mx, \\ af &= aF + \sin nx, \\ (f, g) &= F \cdot \overline{G}. \end{aligned}$$

Each element in the class is determined by a complex number and a non-negative integer; the representation is unique. Postulates 1.0, 1.1, and 1.2 are clearly satisfied. Postulate 1.3 is not satisfied, for let $h = H + \sin px$ and suppose that $[f + (-1)g] + h = h$ for every such h. This implies the relation $F - G + H = H$. We infer that $F = G$, but we can say nothing about n and m. Postulates 1.4–1.8 are satisfied, as are 1.9 and 1.10 also.

EXAMPLE 1.4. Consider the class of infinite one-rowed matrices of complex numbers $x = (x_1, x_2, \cdots)$. No convergence condition is attached. Equality, addition, and multiplication are defined as usual. When x and y are elements of H_0, the inner product is defined as the series

$$(x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

For such elements the inner product is a complex number. Between all other pairs of elements it is defined as the infinite one-rowed matrix $(x_1\bar{y}_1, x_2\bar{y}_2, \cdots)$. Thus the inner product is not

always a complex number. All the postulates save 1.4 are satisfied.

EXAMPLE 1.5. Consider the class H_0, with equality, addition, and multiplication defined as usual. Then, since the series $\sum_{i=1}^{\infty}|x_i|^2$ is convergent,

$$\lim_{n\to\infty}|x_n| = 0,$$

and for each matrix x there exists an integer i such that $|x_i| = \max\{|x_1|, |x_2|, \cdots\}$. Then if k is such that $|y_k| = \max\{|y_1|, |y_2|, \cdots\}$, we define $(x, y) = x_i \cdot \bar{y}_k$. This system satisfies all the postulates save 1.5, as is readily verified.

EXAMPLE 1.6. Consider the class H_0, with equality, addition, and multiplication defined as usual, and define the inner product

$$(x, y) = \max_{i=1,2,\cdots}|y_i|\sum_{k=1}^{\infty}x_k\bar{y}_k.$$

Then clearly postulate 1.6, that of Hermitian symmetry, is not satisfied, whereas the other postulates *are* satisfied.

EXAMPLE 1.7. Consider the class H_0, with equality, addition, and the inner product defined as usual. For multiplication by complex numbers we define

$$a \cdot x = (R(a)x_1, R(a)x_2, \cdots),$$

where $R(a)$ is the real part of a. This system fails on postulate 1.7.

EXAMPLE 1.8. Consider the class H_0, with the definitions as in the Hilbert space of the consistency example, save for the inner product, which is the *negative* of the usual inner product: $(x, y) = -\sum_{i=1}^{\infty}x_i\bar{y}_i$. This leads to a negative-definite quadratic form:

$$(x, x) \leqq 0, (x, x) = 0 \text{ if and only if } x_i = 0, \qquad (i = 1, 2, \cdots).$$

Thus postulate 1.8 is independent.

EXAMPLE 1.9. Consider the class H_0, with equality, addition, and multiplication defined as usual, and the inner product de-

fined to be zero for every pair of elements: $(x, y) \equiv 0$. This obviously satisfies all the postulates but 1.9.

EXAMPLE 1.10. Consider the class H_0, with addition, multiplication, and the inner product defined as usual. For equality we define $x = y$ if and only if

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} |y_i|^2.$$

Then postulates 1.0–1.9 are satisfied, but 1.10 is not, as the following example shows. Let $x = (1, 2, 0, 0, \cdots)$ and $y = (2, 1, 0, 0, \cdots)$. Then $x = y$, for

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} |y_i|^2 = 5.$$

But if $u = (1, 0, 0, \cdots)$ (u is obviously in H_0), we have $(x, u) = 1$, and $(y, u) = 2$.

These eleven examples prove the independence of the postulates.*

5. *Abstract Hilbert Space.* In this paragraph we shall consider the universe of discourse composed of the class K, its relation and operations, subject to the postulates $1.0 - 1.10$, and in addition, the following three.

2.1. *For each positive integer n, there exist elements $x_1, x_2, \cdots, x_n \epsilon K$ such that $a_1x_1 + \cdots + a_nx_n = 0$ if and only if $a_1 = \cdots = a_n = 0$.*

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* The postulates may easily be modified to yield an independent set for a *real* space, that is, a space closed under multiplication by real numbers (the class R). Obvious alterations only are required in §§3–4, and in most of the independence examples. Examples 1.2 and 1.7, however, will not serve, and may be replaced by the following.

EXAMPLE 1.2 (real). Consider the class of infinite one-rowed matrices $x = (x_1, x_2, \cdots)$, where $\sum_{i=1} x_i^2$ is convergent, and the x_i are real, rational numbers. Multiplication by a real number α is defined by $\alpha \cdot x = (\alpha x_1, \alpha x_2, \cdots)$, and the other definitions follow the usual model.

EXAMPLE 1.7 (real). Consider the class of infinite one-rowed matrices of real numbers $x = (x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} x_i^2$ is convergent. Let multiplication by a real number α be defined by $\alpha \cdot x = (N(\alpha)x_1, N(\alpha)x_2, \cdots)$, where $N(\alpha)$ is precisely α if α is an integer, and $N(\alpha)$ is the first integer smaller than α in all other cases.

2.2. *K is separable according to the norm* $(x, x)^{1/2} \equiv \|x\|$.

2.3. *K is complete* (according to the same norm).*

DEFINITION.† When the system of K and its relation and operations satisfies postulates 1.0–1.10 and 2.1–2.3, it is called an abstract Hilbert space, and denoted by H.

It naturally occurs to one to inquire about the independence of the fourteen postulates, viewed as a single system. Before any steps can be taken in this direction, postulates 2.1–2.3 must be restated so as to be intelligible in case some of the other postulates are not satisfied. While this is possible, it leads to many difficulties of expression. I believe that the present treatment is more natural and convenient.

CALIFORNIA INSTITUTE OF TECHNOLOGY

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* J. von Neumann, loc. cit., p. 24. These postulates are identical with the last three of von Neumann. The latter has proved (p. 37) that each of these postulates is independent of the remainder of his set.

† The reader may easily convince himself that this definition is justified, and that our postulate system is equivalent to that of von Neumann. This latter system is known to be categorical. That is, an isomorphism can be established between any two examples of abstract Hilbert space.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

ANN ARBOR, MICHIGAN, SUMMER MEETING AND COLLOQUIUM, September 10-14, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than August 6. Abstracts received by July 6 will appear in the July issue of the BULLETIN. Professor H. S. Vandiver will deliver a series of colloquium lectures on *Fermat's last theorem and related topics in number theory.* By invitation of the Program Committee, Professor G. Y. Rainich will deliver an address entitled *Product integrals*, and Professor G. T. Whyburn an address entitled *On the structure of continua.*

NEW YORK CITY, October 26, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, Low Memorial Library, Columbia University, New York City, not later than September 28. Abstracts received by the Secretaries by September 6 will appear in the September issue of this BULLETIN. By invitation of the Program Committee, Professor J. L. Walsh will deliver an address on *Interpolation by rational functions.*

LEXINGTON, KENTUCKY, November 29-30, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 1. All these will appear in the November issue of this Bulletin. By invitation of the Program Committee, addresses will be delivered by Professor L. L. Dines on *Convex domains and linear inequalities*, and by Professor C. G. Latimer on *The arithmetic of generalised quaternions.*

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 30, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than November 2. All these will appear in the November issue of this BULLETIN.

ST. LOUIS, MISSOURI, ANNUAL MEETING, December 31, 1935-January 3, 1936.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 26, 1935. Abstracts received by the secretaries by November 2 will appear in the November issue of this Bulletin.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or Low Memorial Library, Columbia University, New York City.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, Low Memorial Library, Columbia University, New York City.

WHOLE NUMBER 429

CONTENTS

	PAGE
Tetrahedra Associated with Canonical Expansions for a Curved Surface. By P. O. Bell	353
On a Theorem of Plessner. By Nelson Dunford	356
On the Law of Quadratic Reciprocity. By Albert Whiteman	359
The Arithmetized Expansions for Certain Doubly Periodic Functions of the Third Kind. By G. D. Nichols	361
A Connectedness Theorem in Abstract Sets. By W. M. Whyburn	365
Note on Non-Analytic Functions. By I. M. Sheffer	367
A New Solution of the Gauss Problem on $h(s^2d)/h(d)$. By Gordon Pall	373
On a Reduction of a Matrix by the Group of Matrices Commutative with a Given Matrix. By P. L. Trump	374
A Generalization of Harmonic Functionals. By F. G. Dressel	381
A Note on the Equilibrium Point of the Green's Function for an Annulus. By Deborah M. Hickey	389
Note on a Remark of D. J. Struik on Correlation Coefficients. By Y. B. D. Derksen	394
Note on the Geometric Interpretation of the Vanishing of a Certain Projective Invariant of Two Conics. By Robin Robinson	399
A Group of Operations on a Partially Colored Map. By Irving Kittell	407
On the Equivalence of Two Methods of Defining Stieltjes Integrals. By B. C. Getchell	413
A New Proof of Minkowski's Theorem on the Product of Two Linear Forms. By R. Q. Seale	419
Some Observations on the Theory of Fourier Transforms. By E. Hille, A. C. Offord, and J. D. Tamarkin	427
A Theorem Concerning Locally Peripherally Separable Spaces. By F. B. Jones	437
A Reduced Set of Postulates for Abstract Hilbert Space. By A. E. Taylor	439

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